

Trimer Covers in the Triangular Grid: Twenty Mostly Open Problems

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ABSTRACT. In the past three decades, the study of rhombus tilings and domino tilings of various plane regions has been a thriving subfield of enumerative combinatorics. Physicists classify such work as the study of dimer covers of finite graphs. In this article we move beyond dimer covers to trimer covers, introducing plane regions called *benzels* that play a role analogous to hexagons for rhombus tilings and Aztec diamonds for domino tilings, inasmuch as one finds many (so far mostly conjectural) exact formulas governing the number of tilings.

1. Introduction

If $G = (V, E)$ is a finite graph, a **trimer** on G is a three-element subset of V whose induced subgraph in G is connected, and a **trimer cover** is a partition of V into trimers. Solving the trimer model for G means counting the possible trimer covers. In the physics literature one often assigns weights to the trimers, and solving the weighted model means finding the sum of the weights of all the trimer covers of G , where the weight of an individual trimer cover is the product of the weights of its constituent trimers; this sum is called the *partition function*. As a special case, one can set some of the weights equal to 1 and the rest equal to 0; then the partition function is just the number of trimer covers that only use “permitted” trimers (trimers that have been assigned weight 1).

Physicists have studied the asymptotics of the weighted trimer model on the triangular lattice and obtained formulas for the entropy in various regimes (see [10]). These results can be seen as analogous to formulas for the entropy of dimer models on various kinds of finite graphs. However, as far as I am aware there have not been (up to now) any exact enumerative results for trimer models on finite subgraphs of the triangular lattice in the style of the well-known enumerations of rhombus tilings of hexagons and domino tilings of the Aztec diamond [6].

In this article I introduce finite subgraphs of the triangular lattice that should interest enumerative combinatorialists inasmuch as the number of trimer covers appears to be given by exact formulas in many cases. See Figure 1 which shows both a trimer cover of a finite subgraph of the triangular lattice and the associated tiling of a dual 2-complex whose faces correspond to vertices of the triangular lattice. In

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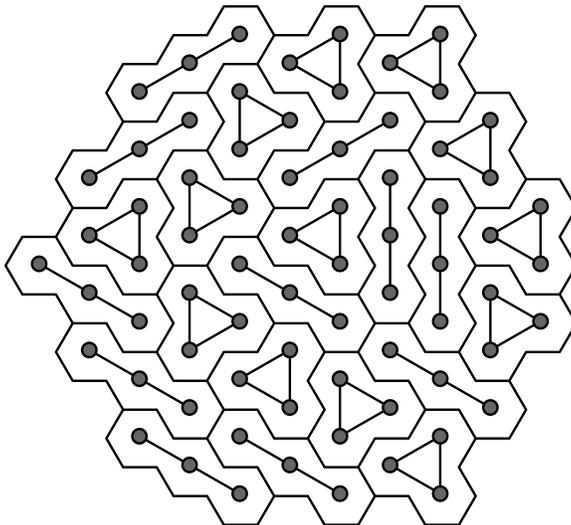


FIGURE 1. The (9,9)-benzel tiled by stones and bones and the associated trimer cover of the (9,9)-benzel graph.

tiling language, the tiled region in the Figure is a *polyhex* and the tiles are *trihexes*. I have dubbed the tiled region a *benzel*, and the allowed tiles *stones* and *bones*. I will switch between the trimer picture and the tiling picture throughout the article. The graph admits an essentially unique coloring with the colors red, blue, and green in which no two adjacent vertices have been assigned the same color; under such a coloring, each stone-trimer and bone-trimer contains exactly one vertex of each color.

This article is a companion to the talk I gave at the Open Problems in Algebraic Combinatorics conference on May 18, 2022; the slides and video can be accessed through the conference website.

Consider the complex plane tiled by unit hexagonal cells centered at 1 , ω , and ω^2 (here and hereafter ω denotes $e^{2\pi i/3}$); the cell centered at α has corners at $\alpha \pm 1$, $\alpha \pm \omega$, and $\alpha \pm \omega^2$. Given positive integers a, b satisfying $2 \leq a \leq 2b$ and $2 \leq b \leq 2a$, we define the **(a, b)-benzel** as the union of the cells that lie fully inside the hexagon with vertices $a\omega + b$, $-a\omega^2 - b$, $a\omega^2 + b\omega$, $-a - b\omega$, $a + b\omega^2$, and $-a\omega - b\omega^2$ (a hexagon centered at 0 with threefold rotational symmetry whose side-lengths alternate between $2a - b$ and $2b - a$, degenerating to a triangle when $a = 2b$ or $b = 2a$), as shown in Figure 2 for $a = 4$, $b = 6$. (Mnemonically, a is the height of the top edge of the hexagon *above* 0 , and b is the depth of the bottom edge of the hexagon *below* 0 , using appropriately scaled units.) We lose no generality in assuming $2 \leq a \leq 2b$ and $2 \leq b \leq 2a$, since the (a, b) -benzel as defined above coincides with the $(a, a - b)$ -benzel when $a > 2b$ and with the $(b - a, b)$ -benzel when $b > 2a$, the former satisfying $a \leq 2(a - b)$ and the latter satisfying $b \leq 2(b - a)$.

Here is an alternative description of benzels in terms of the centers of the cells using barycentric coordinates relative to the triangle with vertices 1 , ω , and ω^2 . Each cell's center point α belongs to $\mathbb{Z}[\omega]$ and can be represented by the unique $(i, j, k) \in \mathbb{Z}^3$ satisfying $i + j\omega + k\omega^2 = \alpha$ and $i + j + k = 1$ (here and hereafter i

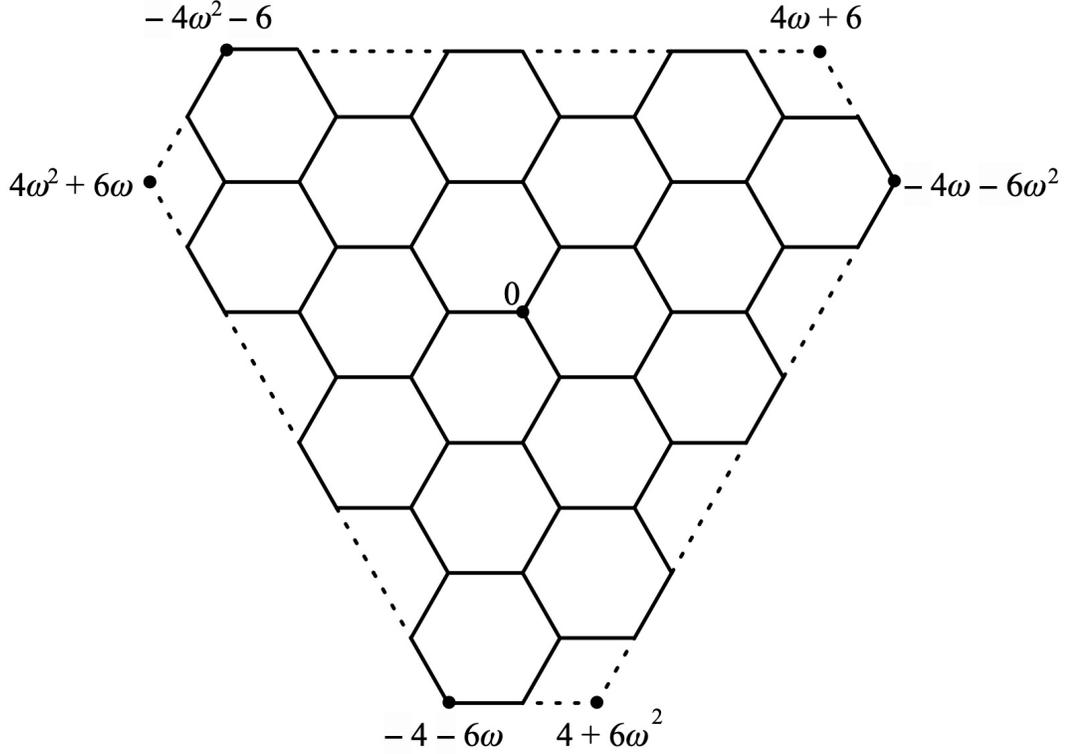


FIGURE 2. The (4,6)-benzel and its enclosing hexagon.

denotes a nonnegative integer as opposed to the square root of -1). Then the (a, b) -benzel consists of those cells whose centers (i, j, k) satisfy $-(a-1) \leq j-i, k-j, i-k \leq b-1$. Figure 3 shows the (4,6)-benzel with its cells marked with the barycentric coordinates of their respective center points.

Let $V = \{(i, j, k) \in \mathbb{Z}^3 : i + j + k = 1, -(a-1) \leq j-i, k-j, i-k \leq b-1\}$. Given (i_1, j_1, k_1) and (i_2, j_2, k_2) in V , join (i_1, j_1, k_1) and (i_2, j_2, k_2) by an edge when $|i_1 - i_2| + |j_1 - j_2| + |k_1 - k_2| = 2$ (that is, when the unit hexagons centered on those two vertices share an edge). This is the **(a, b) -benzel graph**.

The (a, b) -benzel has threefold rotational symmetry but for most a, b it does not have bilateral symmetry. Exchanging a and b corresponds to reflecting the benzel across a horizontal axis (or if one prefers across an axis making a 60 degree angle with the horizontal axis).

We consider tilings of the (a, b) -benzel by way of five sorts of prototiles. These prototiles (shown in Figure 4) are the **right(-pointing) stone**, the **left(-pointing) stone**, the **vertical bone**, the **rising bone**, and the **falling bone**. The right stone is a benzel (specifically, the (2,2)-benzel) but the left stone is not. As usual, the allowed tiles are the translates of the prototiles. Dually we form spanning subgraphs of the (a, b) -benzel graph whose connected components all consist of three vertices. Figure 1 shows a tiling of the (9,9)-benzel and the associated trimer cover of the (9,9)-benzel graph.

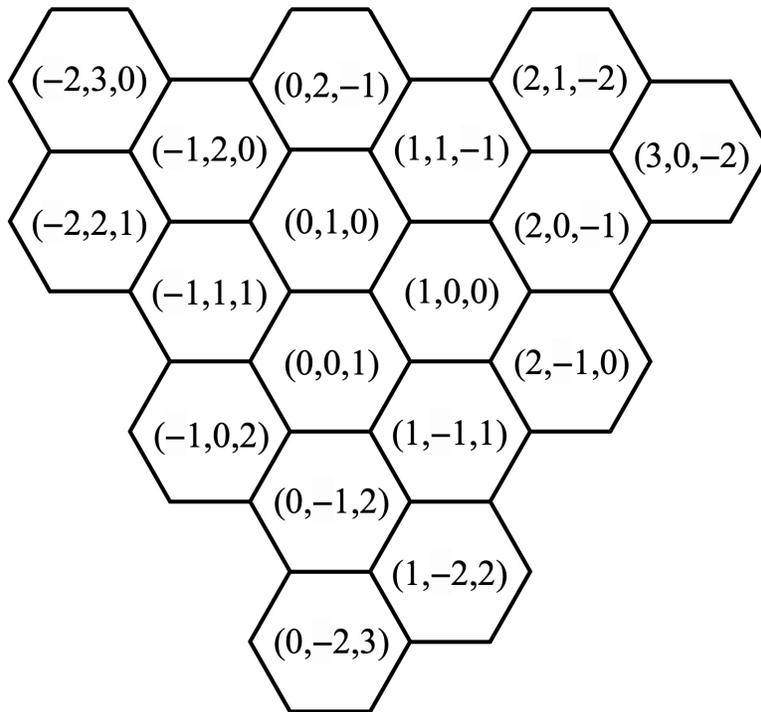


FIGURE 3. Barycentric coordinates for the cells of the (4,6)-benzel.

There is a third kind of trihex that consists of three hexagons whose centers form a 120-degree angle. Timothy Chow has dubbed this kind of tile a *phone*. Since the associated trimers do not respect the tripartite nature of the triangular lattice, I have chosen to exclude those trimers from consideration here. This is not to say that such trimers are unworthy of study. Just as enumeration of dimer covers should not be limited to the bipartite case, enumeration of trimer covers should not be limited to the tripartite case. However, it seems best to start with the problems that are likely to be easier. In particular, the Conway-Lagarias invariant discussed below only applies if all our trihexes are stones and bones.

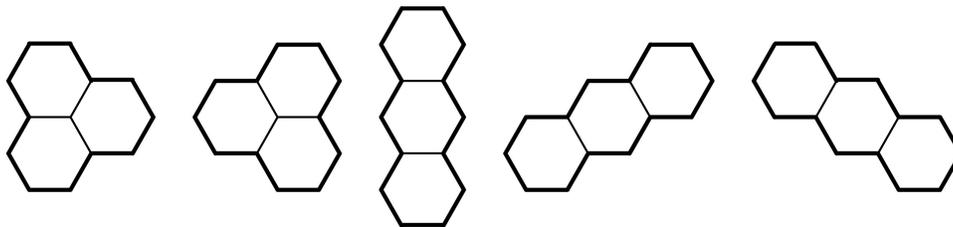


FIGURE 4. The five prototiles: the right stone, the left stone, the vertical bone, the rising bone, and the falling bone.

Conway and Lagarias [2] studied tilings like the ones considered here (calling stones and bones T_2 and L_3 tiles respectively). They showed that for any simply-connected region in the hexagonal grid that can be tiled by stones and bones, the total area of the right stones minus the total area of the left stones does not depend on the specific tiling but only depends on the region being tiled. This is the *Conway-Lagarias invariant* of the region. It can be shown (see the companion article [4]) that the area of the (a, b) -benzel (as measured by the number of tiles) is given by

$$\begin{aligned} &(-a^2 + 4ab - b^2 - a - b)/2 && \text{if } a + b \equiv 0 \text{ or } 2 \pmod{3}, \\ &(-a^2 + 4ab - b^2 - a - b + 2)/2 && \text{if } a + b \equiv 1 \pmod{3} \end{aligned}$$

while the value of the Conway-Lagarias invariant is

$$\begin{aligned} &(3a^2 - 6ab + 3b^2 - a - b)/2 && \text{if } a + b \equiv 0 \pmod{3}, \\ &(-a^2 + 4ab - b^2 - a - b + 2)/2 && \text{if } a + b \equiv 1 \pmod{3}, \\ &(3a^2 - 6ab + 3b^2 + a + b - 2)/2 && \text{if } a + b \equiv 2 \pmod{3}. \end{aligned}$$

(The expressions in Theorem 2 of [4] have opposite sign because the benzels in that article are mirror images of the benzels considered here.) Note that when $a + b \equiv 1 \pmod{3}$, the Conway-Lagarias invariant is equal to the area of the benzel so that the tiling must consist entirely of right-pointing stones; for instance, this is the case with the $(4, 6)$ -benzel shown earlier.

The set of 5 prototiles has $2^5 - 1 = 31$ nonempty subsets, and for each, we can ask in how many ways it is possible to tile the (a, b) -benzel, that is, in how many ways it is possible find translates of the prototiles whose interiors are disjoint and whose union is the benzel. There is some redundancy here. Because the benzel has threefold rotational symmetry, and because 120 degree rotations preserve the two stones' orientation (right versus left), the number of tilings depends only on (a) whether right stones are allowed, (b) whether left stones are allowed, and (c) how many of the three kinds of bones are allowed (0, 1, 2, or 3). Thus there are really only $(2)(2)(4) - 1 = 15$ classes of tiling problems to consider. For $0 \leq i, j \leq 1$ and $0 \leq k \leq 3$ we define $T_{ijk}(a, b)$ as the number of ways to tile the (a, b) -benzel if the set of allowed prototiles contains the right stone iff $i = 1$, contains the left stone iff $j = 1$, and contains k of the bones. We say that such a tiling is an i, j, k tiling.

It is not hard to show that for each of the 15 cases, $T_{ijk}(a, b) = T_{ijk}(b, a)$. It is also not hard to show that the $(n, 2n)$ -benzel is the same as the $(n, 2n - 1)$ -benzel and the $(n, 2n - 2)$ -benzel. Consequently, in the tables that follow I provide values for $T_{ijk}(a, b)$ only for $2 \leq a \leq 2b - 2$ and $2 \leq b \leq 2a - 2$. Lastly, it is not hard to show that when $i = 0$ and $a + b \equiv 2 \pmod{3}$, $T_{ijk}(a, b) = 0$. That is because in this case the Conway-Lagarias invariant is equal to the positive number $(3(a - b)^2 + (a + b - 2))/2$, implying that every tiling of the (a, b) -benzel must have at least one right-pointing stone.

David desJardins wrote a general purpose program `TilingCount` that I used to enumerate tilings of regions with various sets of allowed prototiles. This led to the questions and conjectures that appear below. I am happy to share the code and the data on which my conjectures are based (much of which can be found in the Online Encyclopedia of Integer Sequences). Table 1 is a map of the first eighteen problems presented in this article as they relate to those fifteen cases. Rows describe which stones are allowed; columns describe how many bones are allowed. (This table omits

TABLE 1. A map of the first eighteen problems.

	Two kinds of bones	Three kinds of bones
No stones	(no tilings exist)	type 003: prob. 1
Left stones	type 012: probs. 2–3	type 013: prob. 4
Right stones	type 102: prob. 5	type 103: probs. 6–7
Both kinds of stones	type 112: probs. 8–13	type 113: probs. 14–18

cases where the number of allowed bone prototiles is zero or one; in such situations at most one tiling exists, even when both of the stone prototiles are allowed.)

Benzels behave differently according to whether $a + b$ is 0, 1, or 2 (mod 3), so in what follows we will sometimes subdivide analyses according to the congruence class of $a + b$.

In the cases where only two kinds of bone tiles are permitted, the allowed tilings can be viewed as ribbon tilings, as in [5]; indeed, this was the mode of presentation employed in [2], with four of the five prototiles being depicted as ribbons. Switching over to the ribbon tilings presentation has already yielded solutions to problems 2 and 3 in [3], and is likely to provide leverage on other problems in the first column of the table.

2. No stones, three kinds of bones

Prior to the conference, I was able to show that if an (a, b) -benzel can be tiled by bones alone, then we must have $a = k(3k - 1)/2$ and $b = k(3k + 1)/2$ (or vice versa) for some $k \geq 2$. Several attending students found a proof that this necessary condition is also sufficient. (Note that such benzels belongs to the case $a + b \equiv 0 \pmod{3}$.) Jesse Kim found the most complete solution, providing an explicit proof that the tiling he described works for all k . It appears (see OEIS entry A364134) that the number of such tilings grows exponentially in k^4 .

Problem 1: Find an exact formula for $T_{003}(k(3k - 1)/2, k(3k + 1)/2)$.

Of course, even short of an exact formula, any method of determining the number of tilings that is more efficient than brute-force enumeration (e.g., a recurrence relation) would be of interest.

See [4] for more discussion of no-stones tilings of benzels.

3. Left stones, two kinds of bones

Table 2 shows the values of $T_{012}(a, b)$ for $a, b \leq 10$.

Problem 2: Is it true that $T_{012}(3n, 3n) = 2^n n!$ for $n \geq 1$?

Problem 3: Is it true that $T_{012}(3n + 1, 3n + 2) = 2^n n!$ for $n \geq 1$?

Comment: Colin Defant, Rupert Li, Benjamin Young and I worked on problem 2 during the conference and later succeeded in solving Problems 2 and 3; see [3]. We were also able to verify that the pattern of entries equal to 0 and 1 in Tables 2 through 4 persists for all larger values of a and b .

4. Left stones, three kinds of bones

Table 3 shows the values of $T_{013}(a, b)$ for $a, b \leq 10$.

TABLE 2. Values of $T_{012}(a, b)$.

$a \setminus b$	2	3	4	5	6	7	8	9	10
2	0								
3		2	0						
4		0	0	2	0				
5			2	0	0	0	0		
6			0	0	8	0	0	0	0
7				0	0	0	8	0	0
8				0	0	8	0	0	0
9					0	0	0	48	0
10					0	0	0	0	0

TABLE 3. Values of $T_{013}(a, b)$.

$a \setminus b$	2	3	4	5	6	7	8	9	10
2	0								
3		3	0						
4		0	0	9	0				
5			9	0	0	2	0		
6			0	0	144	0	0	0	0
7				2	0	0	1143	0	0
8				0	0	1143	0	0	825
9					0	0	0	73454	0
10					0	0	825	0	0

When a and b are such that the Conway-Lagarias invariant is strictly positive, the (a, b) -benzel cannot be tiled by bones and left stones; the corresponding entries in the table must be zero. On the other hand, when a and b are such that the Conway-Lagarias invariant is negative or zero, the entries in the table are observed to be positive, though I see no reason for concluding that they are.

Problem 4: Is it true that when the Conway-Lagarias invariant associated with the (a, b) -benzel is negative or zero, tilings of type 013 exist?

5. Right stones, two kinds of bones

Table 4 shows the values of $T_{102}(a, b)$ for $a, b \leq 10$.

Problem 5: Is it true that

$$T_{102}(n + 3k, 2n + 3k - 1) = \prod_{i=1}^k \frac{(2i)!(2i + 2n - 2)!}{(i + n - 1)!(i + n + k - 1)!}$$

for $k \geq 0$ and $n \geq 1$ (except $(k, n) = (0, 1)$)? Equality has been verified for $0 \leq k \leq 5, 1 \leq n \leq 5$.

Comment: This formula and the a, b symmetry relation together provide a conjectural enumeration of tilings of type 102 of the (a, b) -benzel for all a, b satisfying

TABLE 4. Values of $T_{102}(a, b)$.

$a \setminus b$	2	3	4	5	6	7	8	9	10
2	1								
3		0	1						
4		1	2	0	1				
5			0	1	4	0	1		
6			1	4	0	1	10	0	1
7				0	1	8	0	1	28
8				1	10	0	1	24	0
9					0	1	24	0	1
10					1	28	0	1	48

TABLE 5. Values of $T_{103}(a, b)$.

$a \setminus b$	2	3	4	5	6	7	8	9	10
2	1								
3		0	1						
4		1	7	0	1				
5			0	1	33	2	1		
6			1	33	0	1	164	21	1
7				2	1	666	0	1	864
8				1	164	0	1	12430	0
9					21	1	12430	0	1
10					1	864	0	1	655721

$a + b \equiv 2 \pmod{3}$. (When $a + b \equiv 1$, the number of tilings is 0; when $a + b \equiv 2$, the number of tilings is 1.)

Comment: Three special cases merit special attention. When $k = 1$, the right-hand side of the equation is twice the n th Catalan number; when $n = 1$, the right-hand side of the equation is $2^k k!$; and when $n = 2$, the right-hand side of the equation is $2^k (k + 1)!$.

6. Right stones, three kinds of bones

Table 5 shows the values of $T_{103}(a, b)$ for $a, b \leq 10$.

Problem 6: Is it true that $T_{103}(n, 2n - 3) = (3n + 3)(3n - 7)! / (n - 5)!(2n - 1)!$ for $n \geq 5$? (The formula also works for $n = 3$ and $n = 4$ if one treats $1/(-1)!$ and $1/(-2)!$ as 0.) Equality has been verified for $5 \leq n \leq 16$.

Aside from the fact that Problems 2 and 6 involve different prototile sets, the two problems differ in another important way: in problem 2 we have $b/a \rightarrow 1$ as $n \rightarrow \infty$ while in problem 6 we have $b/a \rightarrow 2$ as $n \rightarrow \infty$. In the former case we say that the sequence is associated with a **central diagonal** of the table of values while in the latter case we say that the sequence is associated with a **peripheral diagonal** (recall that a/b and b/a cannot exceed 2).

TABLE 6. Values of $T_{112}(a, b)$.

$a \setminus b$	2	3	4	5	6	7	8	9	10
2	1								
3		2	1						
4		1	4	6	1				
5			6	1	16	22	1		
6			1	16	48	1	68	90	1
7				22	1	224	512	1	304
8				1	68	512	1	3360	6736
9					90	1	3360	15360	1
10					1	304	6736	1	168960

TABLE 7. Values of $T_{112}(3n, 3n)$.

$$\begin{aligned}
& 2^1, \\
& 2^4 3^1, \\
& 2^{10} 3^1 5^1, \\
& 2^{16} 7^1 11^1 13^1, \\
& 2^{28} 3^2 7^1 13^1 17^1, \\
& 2^{38} 3^2 11^1 17^2 19^2, \\
& 2^{50} 5^1 11^2 13^1 17^1 19^2 23^2, \\
& 2^{64} 3^3 5^4 11^1 13^2 19^1 23^3 29^1, \\
& 2^{84} 3^4 5^3 13^2 17^1 23^2 29^3 31^2, \dots
\end{aligned}$$

In parallel with the observations that preceded Problem 4, note that when the Conway-Lagarias invariant of a benzel is strictly negative, the benzel cannot be tiled by bones and right-pointing stones; the corresponding entries in the table must be zero. On the other hand, when a and b are such that the Conway-Lagarias invariant is positive or zero, the entries in the table are observed to be positive, though I see no reason for concluding that they are.

Problem 7: Is it true that when the Conway-Lagarias invariant associated with the (a, b) -benzel is positive or zero, tilings of type 103 exist?

7. Both stones, two kinds of bones

Table 6 shows the values of $T_{112}(a, b)$ for $a, b \leq 10$. Table 7 gives the first few values of $T_{112}(3n, 3n)$ in factored form.

Problem 8: With $T(n)$ denoting $T_{112}(3n, 3n)$, is it true that the second quotient $T(n)T(n+2)/T(n+1)^2$ is equal to

$$\frac{256(2n+3)^2(4n+1)(4n+3)^2(4n+5)}{27(3n+1)(3n+2)^2(3n+4)^2(3n+5)}$$

for all $n \geq 1$? Equality has been verified for $1 \leq n \leq 7$. See OEIS entry A352207.

TABLE 8. Values of $T_{112}(3n + 1, 3n + 1)$.

$$\begin{aligned}
& 2^2, \\
& 2^5 7^1, \\
& 2^{10} 3^1 5^1 11^1, \\
& 2^{17} 7^1 11^1 13^2, \\
& 2^{30} 3^1 13^1 17^2 19^1, \\
& 2^{38} 3^1 11^1 17^2 19^3 23^1, \dots
\end{aligned}$$

TABLE 9. Values of $T_{112}(3n + 1, 3n + 2)$.

$$\begin{aligned}
& 2^1 3^1, \\
& 2^9, \\
& 2^9 3^1 5^1 7^1 11^1, \\
& 2^{25} 3^1 7^1 13^1, \\
& 2^{28} 3^2 11^1 13^1 17^2 19^1, \\
& 2^{50} 3^1 11^1 17^1 19^2 23^1, \\
& 2^{49} 3^2 5^2 11^2 13^2 17^1 19^2 23^3, \\
& 2^{81} 3^3 5^4 13^2 23^2 29^2 31^1, \dots
\end{aligned}$$

Comment: David desJardins found the pattern governing the numbers $T_{112}(n)$, with assistance from Christian Krattenthaler, Greg Kuperberg and other members of the domino listserv. The same is true for Problem 10.

Table 8 gives the first few values of $T_{112}(3n + 1, 3n + 1)$ in factored form.

With $T(n) = T_{112}(3n + 1, 3n + 1)$ with $n \geq 1$, it appears that $T(n)$ has no prime factor greater than or equal to $4n$.

Problem 9: Find a formula governing $T_{112}(3n + 1, 3n + 1)$. See OEIS sequence A364481.

Table 9 gives the first few values of $T_{112}(3n + 1, 3n + 2)$ in factored form.

Problem 10: With $T(n)$ denoting $T_{112}(3n + 1, 3n + 2)$, is it true that $T(n)T(n + 3)/T(n + 1)T(n + 2)$ is equal to

$$\frac{65536(2n + 3)(2n + 5)^2(2n + 7)(4n + 3)(4n + 5)^2(4n + 7)^2(4n + 9)^2(4n + 11)}{729(3n + 2)(3n + 4)^3(3n + 5)^2(3n + 7)^2(3n + 8)^3(3n + 10)}$$

for all $n \geq 1$? Equality has been verified for $1 \leq n \leq 8$. See OEIS entry A364482.

Table 10 gives the first few values of $T_{112}(3n - 1, 3n)$ in factored form.

With $T(n) = T_{112}(3n - 1, 3n)$ with $n \geq 1$, it appears that $T(n)$ has no prime factor greater than or equal to $4n$.

Problem 11: Find a formula governing $T_{112}(3n - 1, 3n)$. See OEIS sequence A364483.

We now switch from central diagonals to peripheral diagonals.

Problem 12: Is it true that $T_{112}(n + 2, 2n + 1)$ is the “ n th large Schröder number” (see sequence A006318 in the OEIS) for all $n \geq 1$? Equality has been verified for $1 \leq n \leq 15$.

TABLE 10. Values of $T_{112}(3n - 1, 3n)$.

$$\begin{aligned}
 &1^1, \\
 &2^4, \\
 &2^5 3^1 5^1 7^1, \\
 &2^{16} 11^1 13^1, \\
 &2^{19} 3^1 7^1 11^1 13^2 17^1, \\
 &2^{39} 3^1 17^2 19^2, \\
 &2^{37} 5^1 11^2 13^1 17^2 19^3 23^2, \dots
 \end{aligned}$$

TABLE 11. Values of $T_{113}(a, b)$.

$a \backslash b$	2	3	4	5	6	7	8	9	10
2	1								
3		3	1						
4		1	10	18	1				
5			18	1	84	142	1		
6			1	84	459	1	724	1266	1
7				142	1	5766	19057	1	6516
8				1	724	19057	1	380597	1077681
9					1266	1	380597	3759277	1
10					1	6516	1077681	1	185961668

Problem 13: Is it true that $T_{112}(n + 2, 2n)$ is the number of “royal paths in a lattice of order n ” (see sequence A006319 in the OEIS) for all $n \geq 1$? Equality has been verified for $1 \leq n \leq 15$.

8. Both stones, three kinds of bones

Finally we come to the most permissive situation: all prototiles are allowed. Table 11 shows the values of $T_{113}(a, b)$ for $a, b \leq 10$.

Problem 14: Is it true that $T_{113} = 1$ when $a + b$ is 1 (mod 3)? (Note that $a + b \equiv 1$ is the situation in which the Conway-Lagarias invariant coincides with the area of the region being tiled, so that all the tiles must be right-pointing stones.)

Comment: This problem is resolved in the affirmative by Theorem 1.1 of [3].

It is disappointing that the data for $a + b \not\equiv 1$ (mod 3) do not suggest exact conjectures. On the other hand, it is intriguing that congruence phenomena occur, analogous to Cohn’s 2-adic continuity theorem proved in [1] and conjectural 2-adic phenomena of a similar kind discussed in [7].

Problem 15: Is $T_{113}(n, 2n - 4)$ 2-adically continuous as a function of $n \geq 5$?

Comment: The sequence appears to be constant mod 2, constant mod 4, 2-periodic mod 8, and 8-periodic mod 16 (with repeating pattern 4, 4, 4, 4, 4, 12, 4, 12). The case $n = 4$ breaks the pattern. See sequence A364416 in the OEIS.

Problem 16: Is $T_{113}(n, 2n - 3)$ 2-adically continuous as a function of $n \geq 4$?

Comment: The sequence appears to be constant mod 2, constant mod 4, 2-periodic mod 8, and 8-periodic mod 16 (with repeating pattern 2, 14, 2, 14, 10, 6, 10, 6). The case $n = 3$ breaks the pattern. See sequence A364417 in the OEIS.

At the OPAC 2022 meeting, David Speyer suggested that one might use a different definition of a trimer, namely, a path consisting of three vertices and two edges. Thus, each stone would correspond to three different trimers according to which 2 of the 3 possible edges one used. Equivalently, one could use the original definition of a trimer, with the proviso that each stone will count with weight 3. The new numbers do not satisfy any nice patterns, with one exception: in the case where all prototiles are allowed, using stones of weight 3 seems to give rise to the same 2-adic continuity phenomenon we saw in Problems 11 and 12.

Let $T_{ijk}(a, b; 3)$ denote the weighted sum of the i, j, k tilings of the (a, b) -benzel as defined near the end of section 1, where a tiling with m stones has weight 3^m .

Problem 17: Is $T_{113}(n, 2n - 4; 3)$ 2-adically continuous as a function of $n \geq 5$?

Comment: The sequence appears to be constant mod 2, constant mod 4, constant mod 8, and 8-periodic mod 16 (with repeating pattern 4, 4, 12, 12, 4, 12, 12, 4). The case $n = 4$ breaks the pattern. See sequence A364418 in the OEIS.

Problem 18: Is $T_{113}(n, 2n - 3; 3)$ 2-adically continuous as a function of $n \geq 4$?

Comment: The sequence appears to be constant mod 2, constant mod 4, 2-periodic mod 8, and 8-periodic mod 16 (with repeating pattern 14, 6, 10, 2, 6, 14, 2, 10). The case $n = 3$ breaks the pattern. See sequence A364438 in the OEIS.

Comment: Given that the number of right stones minus the number of left stones in a tiling of a benzel is the same for all tilings of that benzel, a different way to describe the weighting assigned to tilings in Problems 17 and 18 (equivalent up to a power of 3) is to give weight 1 to left stones and weight 9 to right stones. During the final stages of preparing this article for publication, I realized that one could more generally give weight 1 to left stones and weight m to right stones. The resulting enumerations appear to exhibit p -adic continuity when $m + 1$ is a power of the prime p . Other forms of congruential consistency appear in the data as well; for instance, the data for $m = 9$ appear to satisfy 5-adic continuity. I hope to address such patterns in future work.

9. Miscellaneous

The next question is not enumerative; it is an old structural question that has gone unresolved for decades. There are two natural kinds of “2-flips” that can turn one stones-and-bones tiling into another; the first trades two stones of opposite orientation for two bones, and the second trades a stone and a bone for a stone of the same orientation and a bone of a different orientation; see Figure 5.

Problem 19: Can every tiling of a finite simply-connected region using stones and bones be mutated into every other such tiling by means of a succession of 2-flips?

Comment: It is known that the hypothesis that the region be simply-connected cannot be dropped. It is also known that if one restricts to tilings of type 112 (that is, if one prohibits one of the three orientations of bones), then the claim is true; Sheffield proved an equivalent claim in the context of ribbon tilings [8].

In closing, we turn to the regions that Conway, Lagarias, and Thurston originally studied in papers [2, 9] that inspired much of my work on tilings: triangles

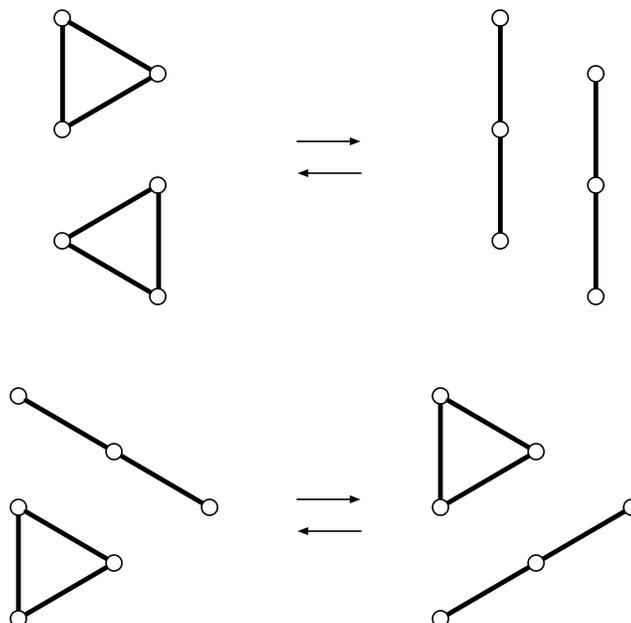


FIGURE 5. 2-flips for changing a trimer cover.

of hexagonal cells, with n cells on each side (“ T_n regions”). Those three authors showed that if one uses stones alone, T_n can be tiled by T_2 ’s (that is, by stones) precisely when n is congruent to 0, 2, 9, or 11 (mod 12). (In our notation, these are tilings of type 110; such tilings were not discussed above since for benzels they are not interesting from an enumerative perspective.)

The question we ask is, how many such tiling are there? Sequence A334875 in the OEIS gives the answers for small values of n . If we look at the prime factorizations of the answers, we notice that the exponent of the prime 2 is creeping upward. Specifically, the multiplicity of the prime 2 in the factorizations of the nonzero terms in this sequence goes 0, 0, 1, 3, 2, 3, 4, 3, 4, 3, 5, 8, 6, 8, \dots . This is a priori surprising, since the probability that a “random” positive integer is divisible by 2^m decreases exponentially as m increases.

Problem 20: As n goes to infinity within the set of natural numbers congruent to 0, 2, 9, or 11 (mod 12), does the number of tiling of T_n by stones converge 2-adically to 0?

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