CASTELNUOVO-MUMFORD REGULARITY AND SCHUBERT GEOMETRY

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ABSTRACT. We study the Castelnuovo-Mumford regularity of tangent cones of Schubert varieties. Conjectures about this statistic are presented; these are proved for the *covexillary* case. This builds on earlier work of L. Li and the author on these tangent cones, as well as work of J. Rajchgot-Y. Ren-C. Robichaux-A. St. Dizier-A. Weigandt and of J. Rajchgot-C. Robichaux-A. Weigandt on the regularity of matrix Schubert varieties.

1. INTRODUCTION

Let GL_n/B be the *complete flag variety*; GL_n is the group of $n \times n$ invertible complex matrices and B is the Borel subgroup of invertible upper triangular matrices. B acts with finitely many orbits $X_w^\circ = BwB/B \cong \mathbb{C}^{\ell(w)}$; $w \in \mathfrak{S}_n$ = the symmetric group on $[n] := \{1, 2, \ldots, n\}$ and $\ell(w)$ is the *Coxeter length* of w, that is, $\ell(w) = \#\{i < j : w(i) > w(j)\}$. Their closures

$$X_w := \overline{X_w^\circ} = \coprod_{v \le w} X_v^\circ$$

are the *Schubert varieties*; here $v \le w$ refers to *(strong) Bruhat order*. Let $T \subset GL_n$ be the maximal torus of invertible diagonal matrices. The T-fixed points are $e_v := vB/B$. To study the local structure of X_w, it suffices to study only the points e_v (for $v \le w$), since B provides local isomorphisms to any other point of $X_v^\circ \subseteq X_w$. A book reference is [7].

Let $(\mathcal{O}_{p,Y}, \mathfrak{m}_p, \Bbbk)$ be the local ring of a point p in a variety Y. The *associated graded ring* [1, Chapter 10] with respect to the $\mathfrak{m}_p$-adic filtration is

$$R_{p,Y} := \mathrm{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Y} = \bigoplus_{i=0}^{\infty} \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1} \quad (\mathfrak{m}_p^0 := \mathcal{O}_{p,Y}).$$

$R_{p,Y}$ has a $\mathbb{Z}$-graded *Poincaré series*

$$\mathrm{PS}_{p,Y}(q) = \sum_{i=0}^{\infty} \dim(\mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}) q^i = \frac{H_{p,Y}(q)}{(1-q)^{\dim(Y)}}, \tag{1}$$

where $H_{p,Y}(q) \in \mathbb{Z}[q]$. $H_{p,Y}(1)$ is the *Hilbert-Samuel multiplicity*. In the case $p = e_v$ and $Y = X_w$, let $\mathrm{PS}_{v,w}(q) = P_{p,Y}(q)$, $R_{v,w} = R_{p,Y}$, and $H_{v,w}(q) = H_{p,Y}(q)$.

We study the *Castelnuovo-Mumford regularity* $\mathrm{Reg}(R_{v,w})$, viewed as a graded module over $\Bbbk[\mathfrak{m}_{e_v}/\mathfrak{m}_{e_v}^2]$. This statistic measures, in some sense, the "complexity" of $R_{v,w}$; see Section 3 for definitions. Outside of Schubert geometry, study of regularity of the associated graded ring appears in, *e.g.*, [3, 23] and the references therein.

Conjecture 1.1. $\mathrm{Reg}(R_{v,w}) = \deg H_{v,w}(q)$.

Conjecture 1.2 (Semicontinuity). *If* $u \le v \le w$ *in Bruhat order then* $\mathrm{Reg}(R_{u,w}) \ge \mathrm{Reg}(R_{v,w})$.

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Conjecture 1.3 (Upper bound). $\mathrm{Reg}(R_{v,w}) \leq \frac{\ell(w)-\ell(v)-1}{2}$.

Proposition 5.4 shows they follow from earlier conjectures with L. Li [16, 17]; see Section 5. Conjectures 1.1 and 1.2 imply that $\mathrm{Reg}(R_{u,v})$ is a singularity measure that falls into the framework of [24]. In particular, it would imply the locus of points $p \in X_w$ with "$\mathrm{Reg}(p) \geq k$" is described using *interval pattern avoidance*.

Speculatively, a strengthening of Conjecture 1.3 holds, namely, $\mathrm{Reg}(R_{v,w}) \leq \deg\, P_{v,w}(q)$ where $P_{v,w}(q)$ is the *Kazhdan-Lusztig polynomial*; but, the evidence is not strong ($n \leq 6$).

The papers [16, 17] study the tangent cones in the case w is *covexillary*, *i.e.*, w *avoids* the pattern 3412 (there are not indices $i_1 < i_2 < i_3 < i_4$ such that $w(i_1), w(i_2), w(i_3), w(i_4)$ are in the same relative order as 3412). This defines a subfamily with a number of prior results. For example, *ibid.* gives formulas for $H_{v,w}(q)$ and related them to the *Kazhdan-Lusztig polynomials*; a combinatorial formula for the latter was already known due to work of A. Lascoux [14]. One also has a "diagonal Gröbner basis theorem" for *matrix Schubert varieties* [13].[1] These results play a role in our work. This is our main result:

Theorem 1.4. *Conjectures 1.1, 1.2, and 1.3 hold if w is covexillary. In this case, there is a combinatorial rule for* $\mathrm{Reg}(R_{v,w})$ *(see Theorem 4.4), and* $\mathsf{Reg}(R_{v,w}) = \deg\, P_{v,w}$.

Our proof of the first part of Theorem 1.4 makes use of [17], which degenerates the tangent cone of the *Kazhdan-Lusztig ideal* $\mathcal{N}_{v,w}$ to the Gröbner limit [13] of the matrix Schubert variety $\overline{X}_{\kappa(v,w)}$ for a *different* covexillary permutation $\kappa(v,w)$. Thereby, $H_{v,w}(q)$ can be expressed in terms of *flagged Grothendieck polynomials* [15, 13]. We were inspired by the paper of J. Rajchgot–Y. Ren–C. Robichaux–A. St. Dizier–A. Weigandt [21], who determine the degree of a *symmetric Grothendieck polynomial* to find the regularity of $\overline{X}_w$ when w is *Grassmannian* (has at most one descent). Ongoing work of J. Rajchgot–C. Robichaux–A. Weigandt [22] extends that formula to vexillary permutations, which we apply.

In Section 2, we recall the notion of Kazhdan-Lusztig ideals/varieties [24]. We also recapitulate necessary results about its tangent cone from [16, 17]. We summarize definitions and facts we need about regularity in Section 3. We then prove our main result in Section 4. Final remarks are collected in Section 5.

2. KAZHDAN-LUSZTIG VARIETIES

Let $\Omega_v^\circ = B_- vB/B$ be the *opposite Schubert cell* where $B_- \subset GL_n$ consists of invertible lower triangular matrices. Ω_{id}° is the *opposite big cell;* it is an affine open neighborhood of $(id)B/B$. Hence $v\Omega_{id}^\circ \cap X_w$ is an affine open neighborhood of X_w centered at e_v. However, by [11, Lemma A.4],

$$X_w \cap v\Omega_{id}^\circ \cong (X_w \cap \Omega_v^\circ) \times \mathbb{A}^{\ell(w)}. \tag{2}$$

Hence it suffices to study the *Kazhdan-Lusztig variety* $\mathcal{N}_{v,w} := X_w \cap \Omega_v^\circ$.

Explicit coordinates and equations for $\mathcal{N}_{v,w}$ were first studied in work with A. Woo [24]. Let $\mathsf{Mat}_{n\times n}$ be the set of all $n \times n$ complex matrices. The coordinate ring is $\mathbb{C}[\mathbf{z}]$ where $\mathbf{z} = \{z_{ij}\}_{i,j=1}^n$ are the functions on the entries of a generic matrix Z. Here z_{ij} corresponds to the entry in the i-th row from the *bottom*, and the j-th column to the right.

[1] Some of these results are stated for *vexillary* rather than covexillary family; this is a matter of convention.

Realize Ω_v° as a affine subspace of $\mathsf{Mat}_{n\times n}$ consisting of matrices $Z^{(v)}$ where $z_{n-v(i)+1,i} = 1$, and $z_{n-v(i)+1,s} = 0, z_{t,i} = 0$ for $s > i$ and $t > n-v(i)+1$. Let $\mathbf{z}^{(v)} \subseteq \mathbf{z}$ be the unspecialized variables. Furthermore, let $Z_{st}^{(v)}$ be the southwest $s \times t$ submatrix of $Z^{(v)}$. The *rank matrix* is

$$r^w = (r_{ij}^w)_{i,j=1}^n$$

(which we index in the same manner), where $r_{ij}^w = \#\{h : w(h) \geq n-i+1, h \leq j\}$. One combinatorial characterization of *Bruhat order* is that $v \leq w$ if and only if $r_{ij}^v \leq r_{ij}^w$ for all $1 \leq i,j \leq n$.

The *Kazhdan-Lusztig ideal* is $I_{v,w} \subset \mathbb{C}[z^{(v)}]$ generated by all $r_{st}^w + 1$ minors of $Z_{st}^{(v)}$ where $1 \leq s,t \leq n$. As explained in [24],

$$\mathcal{N}_{v,w} \cong \mathrm{Spec}\left(\mathbb{C}[z^{(v)}]/I_{v,w}\right);$$

this is reduced and irreducible.

Example 2.1. Let $w = 7314562$, $v = 1423576$ (in one line notation). The rank matrix r^w and the matrix of variables $Z^{(v)}$ are, respectively,

$$r^w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 3 & 4 & 5 & 5 \\ 1 & 1 & 1 & 2 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad Z^{(v)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{61} & 0 & 1 & 0 & 0 & 0 & 0 \\ z_{51} & 0 & z_{53} & 1 & 0 & 0 & 0 \\ z_{41} & 1 & 0 & 0 & 0 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & z_{34} & 1 & 0 & 0 \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & 0 & 1 \\ z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & 1 & 0 \end{pmatrix}$$

The Kazhdan-Lusztig ideal $I_{1423576,7314562}$ contains among its generators, all 2×2 minors of $Z_{25}^{(v)}$ but also inhomogeneous elements such as

$$\begin{vmatrix} z_{51} & 0 & z_{53} \\ z_{41} & 1 & 0 \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = z_{51}z_{33} + z_{53}z_{41}z_{32} - z_{53}z_{31}. \tag{3}$$

This generator, *per se*, does not imply $I_{1423576,7314562}$ is inhomogeneous; however one can confirm the ideal is in fact inhomogeneous with respect to the standard grading using Macaulay2's function `isHomogeneous`. These ideals (and their statistics) can be computed using `https://faculty.math.illinois.edu/~ayong/Schubsingular.v0.2.m2`. □

We also need the *Schubert determinantal ideal* I_w which is defined similarly as $I_{v,w}$ except that we repace $Z^{(v)}$ with the matrix $Z = (z_{ij})$. The zero-set is the *matrix Schubert variety*.

Given $f \in \mathbb{C}[z^{(v)}]$, let $\mathsf{LD}(f)$ denote the lowest degree homogeneous component of f. Now, define the *(Kazhdan-Lusztig) tangent cone ideal* to be

$$I'_{v,w} = \langle \mathsf{LD}(f) : f \in I_{v,w} \rangle.$$

E.g., if f is the polynomial in (3) then $\mathsf{LD}(f) = z_{51}z_{33} - z_{53}z_{31}$. The *tangent cone* of $\mathcal{N}_{v,w}$ is

$$\mathcal{N}'_{v,w} := \mathrm{Spec}\left(\mathbb{C}[z^{(v)}]/I'_{v,w}\right).$$

This can be computed using Macaulay2's `tangentCone` function.

3. CASTELNUOVO-MUMFORD REGULARITY BASICS

The *Castelnuovo-Mumford regularity* of a finitely generated graded module $M = \bigoplus_{j\in\mathbb{Z}} M^{(j)}$ over a standard $\mathbb{N}$-graded ring $S = \bigoplus_{j\geq 0} S^{(j)}$ is defined by

$$\mathrm{Reg}(M) = \max\{f_j(M) + j : j \geq 0\}$$

where

$$f_j(M) := \begin{cases} \sup\{n : H^j_{S_+}(M)_n \neq 0\} & \text{if } H^j_{S_+}(M) \neq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Here $S_+ = \bigoplus_{j>0} S^{(j)}$ is the irrelevant ideal of S and $H^i_{S_+}(M)$ is the i-th local cohomology module of M with respect to S_+ (and its endowed grading). We refer the reader to the book [4, Chapter 15] for further details. One has an expression for the Poincaré series

$$\mathsf{PS}_M(q) = \frac{\mathcal{K}_M(q)}{(1-q)^{\dim(M)}}, \tag{4}$$

where $\mathcal{K}_M(q) \in \mathbb{Z}[q]$; see, *e.g.*, [5, Corollary 4.1.8]. Let $h_M(q)$ be *Hilbert function* and $p_M(q)$ be the *Hilbert polynomial*. Hilbert's theorem states that $h_M(q) = p_M(q)$ for all sufficiently large q. The *postulation number* is

$$\mathrm{post}(M) = \max\{n : h_M(n) \neq p_M(n)\}.$$

By [5, Proposition 4.1.12],

$$\mathrm{post}(M) = \deg\, \mathcal{K}_M(q) - \dim\, M.$$

It is known (and not hard) that when M is Cohen-Macaulay, $\mathrm{Reg}(M) = \mathrm{post}(M) + \dim\, M$. Hence

$$\mathrm{Reg}(M) = \deg \mathcal{K}_M(q). \tag{5}$$

Now suppose $S = \mathbb{C}[x_1, \ldots, x_N]$ and $M = S/J$ is the S module where $J \subseteq S$ is an ideal that is standard graded homogeneous. $M = S/J$ has a minimal free resolution

$$0 \to \bigoplus_j S(-j)^{\beta_{i,j}(S/J)} \to \bigoplus_j S(-j)^{\beta_{i-1,j}(S/J)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(S/J)} \to S/J \to 0.$$

Here $i \leq N$ and $S(-j)$ is the free S-module where degrees of S are shifted by j. Also,

$$\mathrm{Reg}(M) := \max\{j - i : \beta_{i,j}(M) \neq 0\},$$

and

$$\mathsf{PS}_{S/J}(q) = \frac{\mathcal{K}_{S/J}(q)}{(1-q)^N},$$

where $\mathcal{K}(S/J, q) \in \mathbb{Z}[q]$. If S/J is Cohen-Macaulay, (5) says

$$\mathrm{Reg}(S/J) = \deg \mathcal{K}(S/J, q) - \mathrm{ht}_S(J), \tag{6}$$

where $\mathrm{ht}_S(J)$ is the *height* of the ideal J in S. In our application, the algebraic set $V(J)$ is radical and equidimensional; $\mathrm{ht}_S(J)$ is the codimension of the variety $V(J) \subseteq \mathbb{C}^N$.

Example 3.1. Continuing Example 2.1, using Macaulay2's `resolution` and `betti` one can compute the Betti numbers for the minimal free resolution of $T_{1423576,7314562}$ as

	0	1	2	3	4	5	6	7	8	9	10
total:	1	12	61	176	322	392	322	176	61	12	1
0:	1	7	21	35	35	21	7	1	.	.	.
1:	.	5	40	140	280	350	280	140	40	5	.
2:	.	.	.	1	7	21	35	35	21	7	1

In Macaulay2 format, the entry in row j and column i is $\beta_{i,i+j}$. So $\mathrm{Reg}(\mathbb{C}^{(\nu)}/T_{1423576,7314562}) = 2$ is the largest row index of this table. Similarly one checks that $\mathrm{Reg}(\mathbb{C}^{(\nu)}/T_{1234567,7314562}) = 3$, in agreement with Conjecture 1.2. $\square$

4. Proof of Theorem 1.4

4.1. Proof of Conjectures 1.1, 1.2, 1.3 in the covexillary case. Let $R'_{v,w} := \mathbb{C}[\mathbf{z}^{(v)}]/I'_{v,w}$. We claim

$$\mathrm{Reg}(R'_{v,w}) = \deg H_{v,w}. \tag{7}$$

By [16, Theorems 3.1 and 5.5], $\mathrm{Spec}\, R'_{v,w}$ Gröbner degenerates to $\mathrm{init}_{\prec}\overline{X}_{\kappa(v,w)}$ (up to a permutation of coordinates), the Gröbner limit in [13] of a matrix Schubert variety $\overline{X}_{\kappa(v,w)}$ of the covexillary permutation $\kappa(v,w)$. We will define $\kappa(v,w)$ in Section 4.2. At this moment, it suffices to know that $\mathrm{init}_{\prec}\overline{X}_{\kappa(v,w)}$ is a reduced union of coordinate subspaces, whose associated Stanley-Reisner simplicial complex is homeomorphic to a shellable ball or sphere [13, Theorem 4.4]. Shellable simplicial complexes are Cohen-Macaulay, which by definition, means the said union of coordinate subspaces is Cohen-Macaulay [19, Section 13.5.3]. Therefore $\overline{X}_{\kappa(v,w)}$ is Cohen-Macaulay, and hence $\mathrm{Spec}\, R'_{v,w}$ is also Cohen-Macaulay as it also Gröbner degenerates to it [6, Section 15.8].

In [16], one has

$$\mathcal{K}(R'_{v,w}, q) = \frac{H_{v,w}(q)(1-q)^{\ell(w_0 w)}}{(1-q)^{\ell(w_0 v)}}.$$

Thus by (6), $\mathrm{Reg}(R'_{v,w}) = \deg\ H_{w,v}(q) + \ell(w_0 w) - \ell(w_0 w)$, since $\mathrm{ht}_{\mathbb{C}[\mathbf{z}^{(v)}]} I'_{v,w} = \ell(w_0 w)$ (here we use the fact that the tangent cone of $\mathcal{N}_{v,w}$ has the same dimension as $\mathcal{N}_{v,w}$ itself, namely $\ell(w) - \ell(v)$, and that). Thus (7) holds.

Since the tangent cone of $\mathcal{N}_{v,w}$ is $\mathrm{Spec}\, R'_{v,w}$ it follows from (2) that

$$\text{tangent cone } (v\Omega^{\circ}_{\mathrm{id}} \cap X_w) \cong \mathrm{Spec}\, R'_{v,w} \times \mathbb{A}^{\ell(v)}.$$

The tangent cone of any affine open neighborhood of $p \in Y$ is isomorphic to $R_{p,Y}$; see, *e.g.*, [6, Section 5.4] and [20, III.3]. Hence the Cohen-Macaulayness of $R'_{v,w}$ implies the same of $R_{v,w}$, since this property of an affine variety is preserved under cartesian product with affine space. Hence Conjecture 1.1 holds in this case by (4).

Conjecture 1.2 holds in our case since it is shown in [17] that $H_{v,w}(q)$ is semicontinuous. Also, in the covexillary case, one has from *ibid.* that $\deg H_{v,w}(q) = \deg P_{v,w}(q)$ where $P_{v,w}(q)$ is the *Kazhdan-Lusztig polynomial*. By definition $\deg P_{v,w}(q) \leq \frac{\ell(w)-\ell(v)-1}{2}$; this is Conjecture 1.3.

4.2. Permutation combinatorics and the formula. We recall some standard permutation combinatorics; our reference is [18] (although our conventions are upside down from theirs). The *graph* of $w \in \mathfrak{S}_n$ places a $\bullet$ in position $(w(i), i)$ (written in matrix notation).

Cross out all boxes weakly right and weakly above a •; the remaining boxes of $[n] \times [n]$ form the *Rothe diagram* of w, denoted $D(w)$. That is,

$$D(w) = \{(i,j) \in [n] \times [n] : i > w(j), j < w^{-1}(i)\}.$$

The vector $\mathsf{code}(w) = (c_n, c_{n-1}, \ldots, c_1)$ where c_i is the number boxes of $D(w)$ in row i. The *essential set* $E(w)$ of w consists of those maximally northeast boxes of any connected component of $D(w)$, *i.e.*,

$$E(w) = \{(i,j) \in D(w) : (i-1,j), (i,j+1) \notin D(w)\}.$$

Example 4.1. Continuing our running example, where $w = 7314562$, diagram is graphically depicted in Figure 1. Hence

$$D(w) = \{(2,3), (4,2), (4,3), (5,2), (5,3), (5,4), (6,2), (6,3), (6,4)\}$$

and

$$E(w) = \{\mathfrak{e}_1 = (6,5), \mathfrak{e}_2 = (5,4), \mathfrak{e}_3 = (4,2), \mathfrak{e}_4 = (2,3)\}.$$

Moreover, $\mathsf{code}(w) = (0,4,3,2,0,1,0)$.

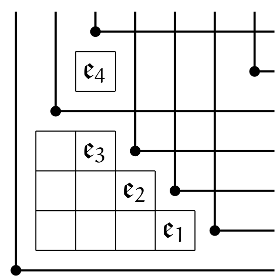
$D(w) =$ 

FIGURE 1. The diagram and essential set for $w = 7314562$.

A permutation in $\mathfrak{S}_n$ is uniquely identified by the values of the rank matrix (r^w_{ij}) when restricted to $D(w)$ or even merely $E(w)$.

Throughout the remainder of this subsection, we assume w is covexillary.

Let $\lambda(w)$ be the partition obtained by sorting $\mathsf{code}(w)$. It is useful to know the *graphical construction* of $\lambda(w)$: Since $(a,b), (c,d) \in E(w)$ then one is weakly northwest of the other [18], it follows there is a unique Young diagram (in French notation) obtained by pushing all boxes of $D(w)$ on a given antidiagonal to the southwest; that is the diagram of $\lambda(w)$.

Example 4.2. Our running example $w = 7314562$ is covexillary with $\lambda(w) = (4,3,2,1)$. □

Given $v \leq w$, [16] defines (and proves the existence of) a different covexillary permutation $\kappa(v,w)$. This is the unique permutation whose essential set is obtained by moving each $\mathfrak{e} = (i,j) \in E(w)$ southwest along its antidiagonal by r^v_{ij} squares to $\mathfrak{e}'$ and imposing that $r^{\kappa(v,w)}_{\mathfrak{e}'} = r^w_{ij} - r^v_{ij}$. By construction, $\lambda(w) = \lambda(\kappa(v,w))$. The graphical construction $\lambda(\kappa(v,w))$ induces a bijection of boxes: $\phi : \lambda(\kappa(v,w)) \to D(\kappa(v,w))$. Define a filling of each box $b \in \lambda(\kappa(v,w))$ with $r^w_{\phi(b)}$. We call this $\mathsf{RRW}(v,w)$, as its provenance is from [22].

Example 4.3. One can check that $\kappa(1423576, 7314562) = 3472561$. □

The next result is the combinatorial rule of Theorem 1.4. It uses a similar result of J. Rajchgot-C. Robichaux-A. Weigandt [22]:

Theorem 4.4.

$$\mathrm{Reg}(R_{v,w}) = \mathrm{Reg}(R'_{v,w}) = \deg H_{v,w} = \sum_{k\geq 1} \sum_{\alpha \in \mathsf{Connected}(\lambda(\kappa(v,w))_{\geq k})} \mathsf{maxdiag}(\alpha), \tag{8}$$

where:

- $\lambda(\kappa(v,w))_{\geq k}$ *is the shape of the subtableau of* $\mathsf{RRW}(v,w)$ *that have entries* $\geq k$*;*
- $\mathsf{Connected}(\kappa(v,w))_{\geq k})$ *are the connected components of the aforementioned shape; and*
- $\mathsf{maxdiag}(\alpha)$ *is the largest northwest-southeast diagonal that appears in* α*.*

Example 4.5. To complete our running example,

$$\mathsf{RRW}(1423576, 7314562) = \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 0 & 0 & & \\ \hline 0 & 0 & 1 & \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array}$$

and hence Theorem 4.4 asserts $\mathrm{Reg} = 2$ (the longest diagonal appearing in the unique 1's component), in agreement with Example 3.1. □

For any $u \in \mathfrak{S}_n$ let $\mathfrak{G}_w(x_1, \ldots, x_n)$ be the *Grothendieck polynomial* [15]. By definition, $\mathfrak{G}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ where w_0 is the longest element in $\mathfrak{S}_n$. If $\ell(us_i) > \ell(u)$ where $s_i = (i\ i+1)$ is a simple transposition, then $\mathfrak{G}_u = \pi_i(\mathfrak{G}_{us_i})$ where

$$\pi_i : \mathbb{Z}[x_1, x_2, \ldots, x_n] \to \mathbb{Z}[x_1, x_2, \ldots, x_n]$$

is the *isobaric divided difference operator* defined by

$$\pi(f) = \frac{(1-x_{i+1})f(\cdots, x_i, x_{i+1}, \cdots) - (1-x_i)f(\cdots, x_{i+1}, x_i, \cdots)}{x_i - x_{i+1}}.$$

4.3. **Proof of Theorem 4.4.** By [16, Theorem 6.6],

$$\mathsf{PS}_{v,w}(q) = \frac{G_\lambda(q)}{(1-q)^{\binom{n}{2}}}, \tag{9}$$

where $G_\lambda(q) = \mathfrak{G}_{w_0\kappa(v,w)}(1-q, 1-q, \ldots, 1-q)$. Comparing (9) with (1) and using the fact that $\dim(X_w) = \ell(w)$, we see that

$$\deg\ H_{v,w} = \deg\ \mathfrak{G}_{w_0\kappa(v,w)} - \left(\binom{n}{2} - \ell(w)\right). \tag{10}$$

On the other hand, since $\lambda(\kappa(v,w)) = \lambda(w)$, one has $\ell(\kappa(v,w)) = \ell(w)$, and hence

$$\ell(w_0\kappa(v,w)) = \binom{n}{2} - \ell(w). \tag{11}$$

Moreover since $\kappa(v,w)$ is covexillary, $w_0\kappa(v,w)$ is vexillary (avoids 2143). The formula of J. Rajchgot-C. Robichaux-A. Weigandt [22] shows (in our conventions) that for any vexillary $u \in \mathfrak{S}_n$ that

$$\deg \mathfrak{G}_u = \ell(u) + \sum_{k\geq 1} \sum_{\alpha \in \mathsf{Connected}(\lambda(w_0 u)_{\geq k})} \mathsf{maxdiag}(\alpha). \tag{12}$$

Hence the theorem follows by combining (10), (11) and (12) with $u = w_0\kappa(v,w)$. □

In general, there are no simple formulas to compute the degree of a Kazhdan-Lusztig polynomial $P_{v,w}(q)$ (we refer the reader to [2, Chapter 5]). This proves the final assertion of Theorem 1.4.

Corollary 4.6. *Let $w \in \mathfrak{S}_n$ be covexillary, then $\deg P_{u,v}$ is computed by the rule of Theorem 4.4.*

Proof. [17, Theorem 1.2] shows $\deg H_{v,w}(q) = \deg P_{v,w}(q)$ when w is covexillary. Now apply Theorem 4.4. $\square$

5. Further results and discussion

These conjectures were asserted in [17]:

Conjecture 5.1. *$R_{v,w}$ is Cohen-Macaulay. Consequently, $H_{v,w} \in \mathbb{N}[q]$.*

That X_w is Cohen-Macaulay does not imply Conjecture 5.1. In fact, C. Huneke [10] established $R_{p,Y}$ being Cohen-Macaulay implies the same for $(\mathcal{O}_{p,Y}, \mathfrak{m}_p, \Bbbk)$, and gave counterexamples for the converse. This is a strengthening of Conjecture 5.1:

Conjecture 5.2 (Semicontinuity). *If $u \leq v \leq w$ then $[q^t]H_{u,w} \geq [q^t]H_{v,w}$.*

Conjecture 5.3 ([17, Proposition 2.1]). $\deg H_{v,w} \leq \frac{\ell(w)-\ell(v)-1}{2}$.

Proposition 5.4. *Conjectures 5.1, 5.2, and 5.3 imply Conjectures 1.1, 1.2, and 1.3.*

Proof. The Cohen-Macaulay assertion of Conjecture 5.1 implies Conjecture 1.1 by the reasoning in our proof of Theorem 1.4. Combined with Conjecture 5.2 gives Conjecture 1.2. Separately, combined with Conjecture 5.3 one would obtain Conjecture 1.3. $\square$

During the preparation of [17], Conjectures 5.1 and 5.3 were checked for $n \leq 7$. Conjecture 5.2 was checked for at least $n \neq 6$ and much of $n = 7$.

Let $\mathrm{maxReg}(n) = \max_{v \leq w \in \mathfrak{S}_n} \mathrm{Reg}(R_{v,w})$.

Conjecture 5.5. $\mathrm{maxReg}(n) = \Theta(n^2)$.

Computational data was not directly useful to arrive at Conjecture 5.5. For $n = 4, 5, 6, 7$, $\mathrm{maxReg}(n) = 1, 2, 3, 5$, respectively. For example, when $n = 7$ the maximizer is the (non-covexillary) $w = 6734512$ at $v = id$. Here $I_{v,w}$ is inhomogeneous and

$$H_{id,6734512}(q) = 1 + 4q + 9q^2 + 9q^3 + 4q^4 + q^5.$$

Let $\overline{\mathrm{maxReg}(n)} = \max_{v \leq w \in \mathfrak{S}_n, w \text{ covexillary}} \mathrm{Reg}(R_{v,w})$. We apply Theorem 4.4 to prove the covexillary case of Conjecture 5.5.

Proposition 5.6. $\overline{\mathrm{maxReg}(n)} = \Theta(n^2)$.

Proof. For the lower bound, first suppose $n = 3j-1$ for $j \geq 1$. Let $v = id$ and $w \in \mathfrak{S}_n$ be the unique permutation with $\mathsf{code}(w) = (1, 2, 3, \ldots, j, 0, 0, \ldots, 0)$. Then w is covexillary, with $\lambda(w) = (j, j-1, \ldots, 3, 2, 1)$. For example, if $j = 4$ then $w = 7, 11, 6, 10, 5, 9, 4, 8, 3, 2, 1$. By our assumption, $\kappa(id, w) = w$. Hence $\mathsf{RRW}(\kappa(id, w))$ is the staircase $\lambda(w)$ where column c from the left is filled by $(c-1)$'s. In our example,

$$\mathsf{RRW}(\kappa(id, w)) = \begin{array}{|c|c|c|c|} \hline 0 & & & \\ \hline 0 & 1 & & \\ \hline 0 & 1 & 2 & \\ \hline 0 & 1 & 2 & 3 \\ \hline \end{array}.$$

Hence, Theorem 4.4 asserts that $\mathrm{Reg}(R_{\mathrm{id},w}) = (j-1)+(j-2)+\ldots+2+1 = \binom{j}{2}$. Now, if $n = 3j$ or $n = 3j+1$, use the same construction as for $n = 3j-1$, except that $\mathsf{code}(w)$ will have an additional 0 or $0, 0$ postpended, respectively. In those two cases, the same analysis implies $\mathrm{Reg}(R_{\mathrm{id},w}) = \binom{j}{2}$. Hence $\overline{\mathrm{maxReg}(n)} = \Omega(n^2)$ follows.

For the upper bound, since $w \in \mathfrak{S}_n$, $\lambda(\kappa(v, w)) \subseteq n \times n$ and $\mathsf{RRW}(\kappa(v, w))$ only uses labels $k \in [n]$. For each such k, the inner sum of (8) contributes $\leq n$. Hence $\mathrm{Reg}(R_{v,w}) \leq n^2$. Therefore, $\overline{\mathrm{maxReg}(n)} = O(n^2)$, as required. □

Corollary 5.7. *Conjecture 1.3 implies Conjecture 5.5.*

Proof. The lower bound of Conjecture 5.5 is immediate from Proposition 5.6. If Conjecture 1.3 holds, then $\mathrm{Reg}(R_{v,w}) \leq \frac{\ell(w)-\ell(v)-1}{2} \leq \ell(w_0) = \binom{n}{2}$. □

Sometimes, $I_{v,w}$ is homogeneous with respect to the standard grading; see [25] and the references therein. In those cases, trivially, $I'_{v,w} = I_{v,w}$ and Cohen-Macaulayness of $I_{v,w}$ and Conjecture 1.1 is automatic. As argued in [17], the covexillary case is interesting precisely because $I'_{v,w} = I_{v,w}$ need not hold in general (as in the case of our running example).

It is also natural to expect that our regularity conjectures are true for other Lie types. We remark that in the minuscule case studied by [8], it is again true that the Schubert varieties admit a dilation action of $\mathbb{C}^*$ and hence the analogue of Conjecture 1.1 holds for a similar reason as in the previous paragraph. This problem should be in reach:

Problem 5.8. *Determine the regularity of tangent cones of Schubert varieties for minuscule* G/P.

We also mention that the *banner permutations* of Z. Hamaker-O. Pechenik-A. Weigandt [9] extend the vexillary permutations and have a description of the Gröbner basis (also, see a further extension by P. Klein [12]). It would therefore be interesting to see if the results of this paper (or of [16, 17]) extend to that setting.

With regards to Theorem 4.4, one can use any rule that computes $\deg(\mathfrak{G}_u)$. Another rule applicable to arbitrary $u \in \mathfrak{S}_n$ has been found by O. Pechenik-D. Speyer-A. Weigandt. On the one hand, the tableau rule of [22] is fitting with the covexillary combinatorics we use. On the other hand, one wonders if that general rule can be adapted to compute $\mathsf{Reg}(R_{u,v})$? We also remark that both of these formulas can be regarded as solving a special case of our regularity problem; see [25, Corollary 2.6] and its proof.

Finally, the Gröbner basis of [16] only uses ± 1 coefficients. Consequently, $H_{v,w}(q)$, and thus Theorem 1.4 is independent of characteristic. Is this true for general $w \in \mathfrak{S}_n$?

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