TRIANGULAR ICE: COMBINATORICS & LIMIT SHAPES

(PDF + E. Guitter IPHT Saclay)

+ B. Debin UC Louvain

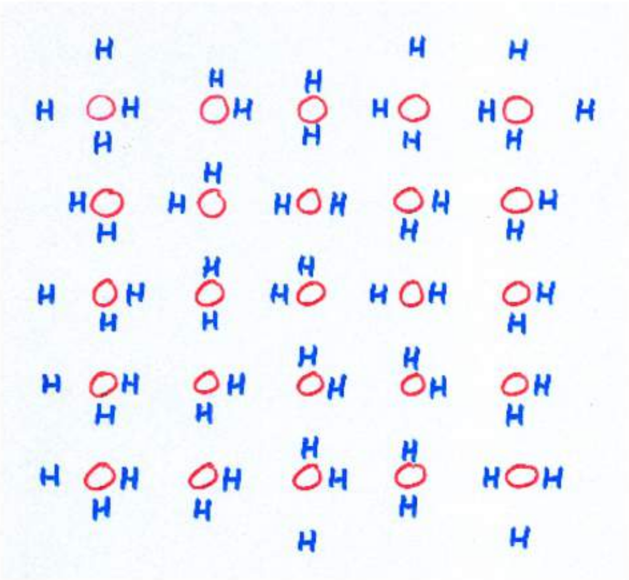
1. ASM, square ice, 6V model and integrability
2. Triangular ice, DWBC, and APM
3. Domino Tilings of the Holey Aztec Square
4. Proof of the APM – HASDT correspondence
5. Combinatorial Conjectures
6. Limit shape / Arctic Phenomenon
7. Conclusion

A Tale of **3** sequences

$$\begin{cases} 1, 3, 23, 433, 19705, 2151843, \ldots \\ 1, 3, 29, 901, 89893, 28793575, \ldots \\ 1, 4, 60, 3328, 678912, 508035072, \ldots \end{cases}$$

OPAC 22

ASM and Square Ice



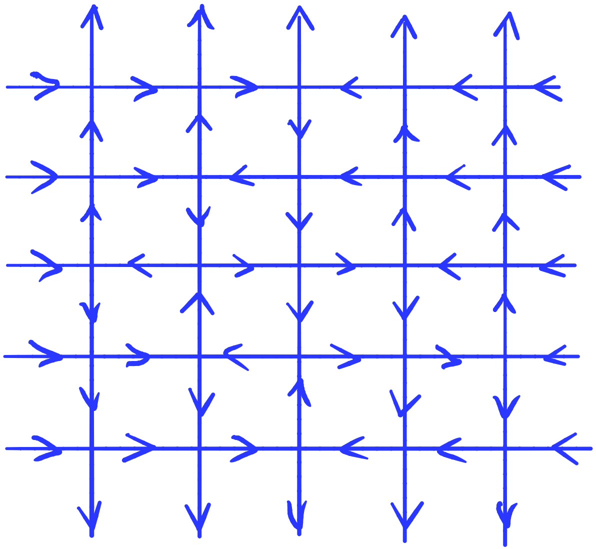
Replace data by dipolar momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

Ice Rule at each vertex

incoming arrows

= # outgoing arrows $\Rightarrow$ 6V

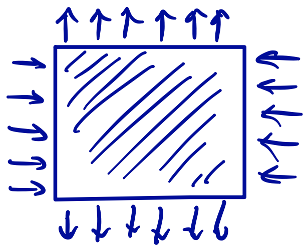
ASM and 6V model



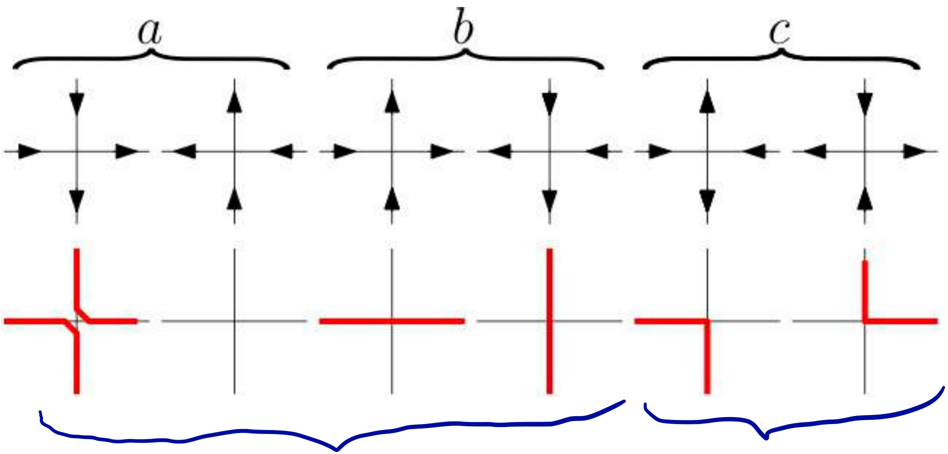
Replace data by dipolar momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

Ice Rule at each vertex
incoming arrows
= # outgoing arrows $\Rightarrow$ 6V

+ Domain Wall Boundary Conditions
(n×n square)



Bijections



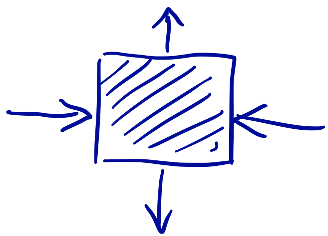
1. 6V configs
2. osculating paths (NW → SE)
3. ASM entries

Transmitter vertices — Reflector vertices

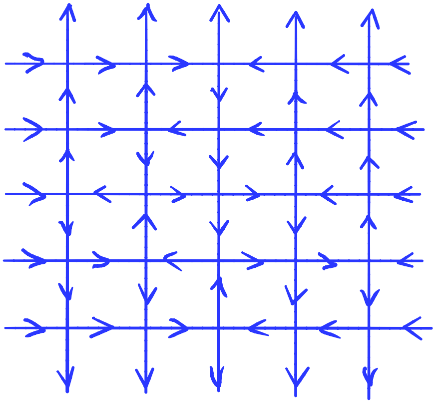
↓ ↓ ↓ ↓ ↓ ↓

0 0 0 0 +1 −1

Alternance conditions

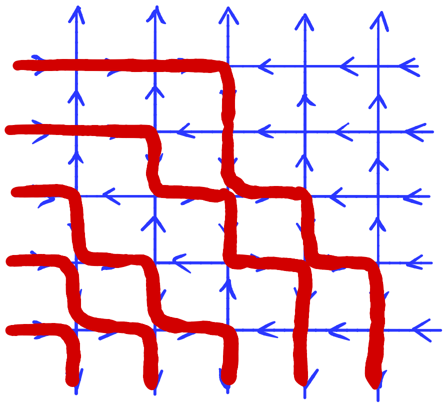


odd # of reflections!



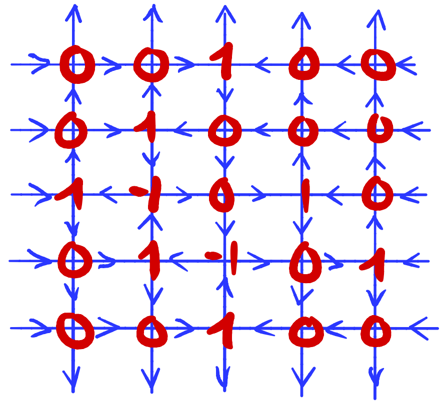
6V

①



Osculating Paths

②



ASM

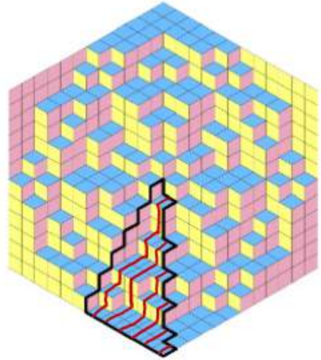
③

n

$$\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&1&0&-1&1\\ 0&0&0&1&0&0\\ 0&1&-1&0&1&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \quad n$$

ASM

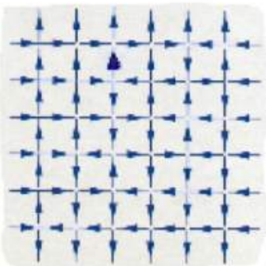
$$ASM_n = \frac{\prod_{i=0}^{n-1}(3i+1)!}{\prod_{i=0}^{n-1}(n+i)!}$$



$2n$

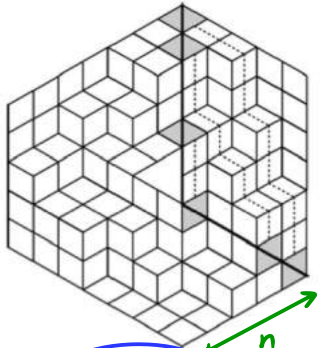
TSSCPP

n



n

6V DWBC



$n+2$

n

$\frac{2\pi}{3}$

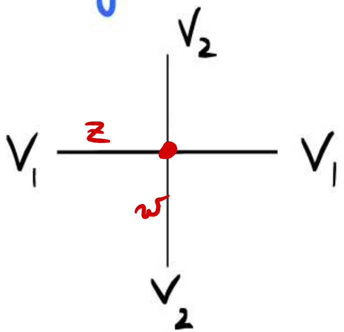
DPP

Korepin, Izergin
Andrews
Kuperberg, Zeilberger
Razumov, Stroganov
Cantini Sportiello
Zinn-Justin, PDF, Behrend
DeGier, Nienhuis
Knutson, Krattenthaler, Fisher

INTEGRABILITY

- Boltzmann weights

R operator



$$R(z,w): V_1 \otimes V_2 \to V_2 \otimes V_1$$

$\dim V_i = 2$

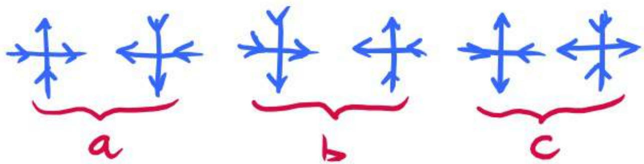
$V_1 = \langle \rightarrow, \leftarrow \rangle \ni \alpha$

$V_2 = \langle \uparrow, \downarrow \rangle \ni \beta$

<u>matrix entries</u> in $\alpha \otimes \beta \to \beta \otimes \alpha$

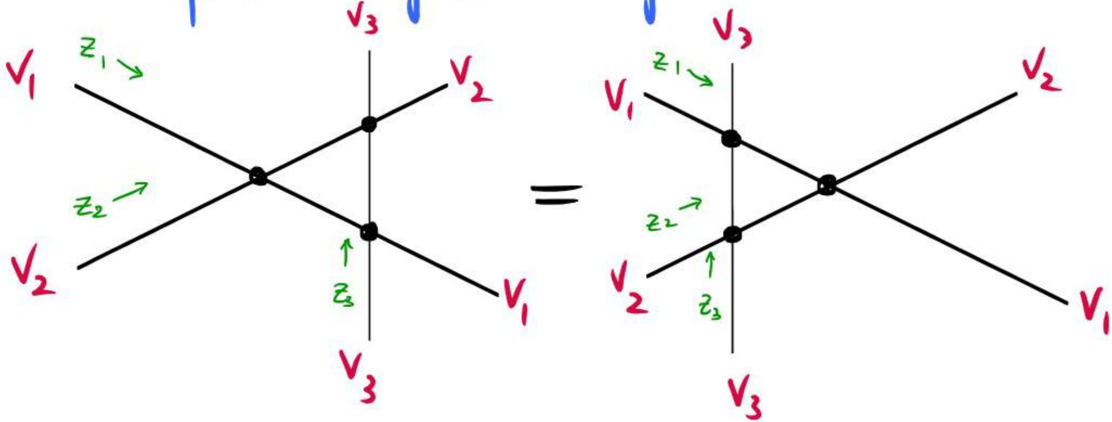
6 non-zero entries out of 16

(ice rule).



YANG-BAXTER RELATION

One can pick "integrable weights" such that:



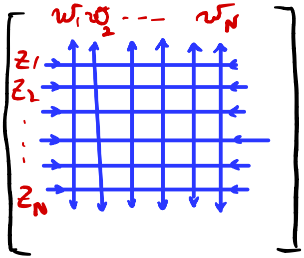
(a cubic identity for R operators
from $V_1 \otimes V_2 \otimes V_3$ to $V_3 \otimes V_2 \otimes V_1$

$$\boxed{a(z,w) = z - w \; ; \; b(z,w) = q^{-2}z - q^{2}w \; ; \; c(z,w) = (q^{2} - q^{-2})\sqrt{zw}}$$

$\dim V = 2$; TRIGONOMETRIC R-matrix of 6V MODEL

IZERGIN-KOREPIN Determinant

Z_{6V}

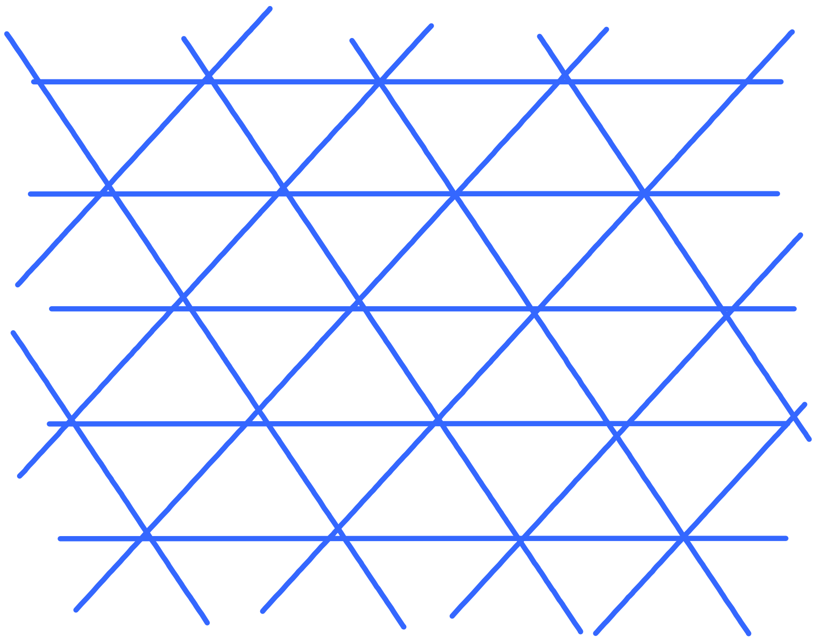


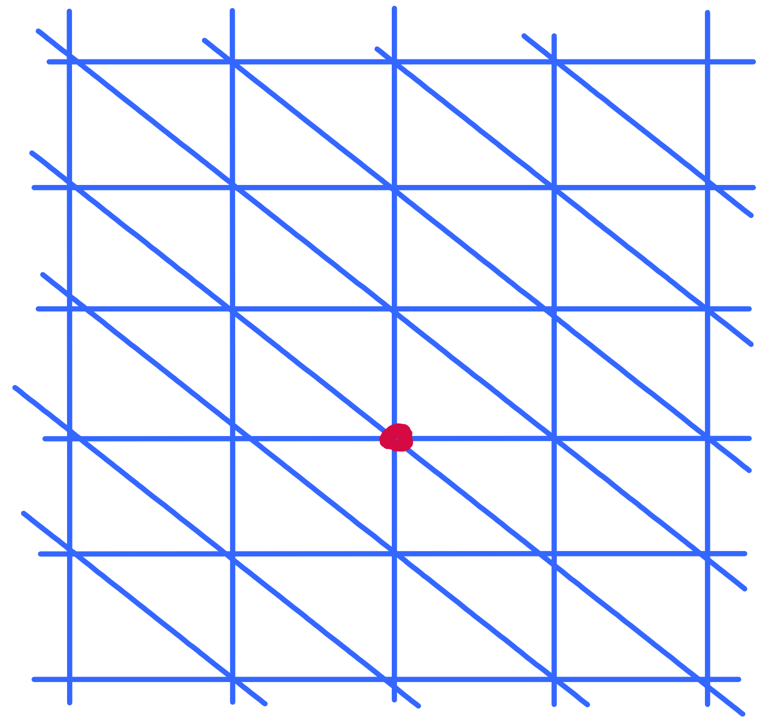
$$= \frac{\prod_{i,j=1}^{N} a(z_i,w_j)\, b(z_i,w_j)}{\prod_{1\le i<j\le N} (z_i-z_j)(w_i-w_j)} \times$$

$$\times \det\left(\frac{c(z_i,w_j)}{a(z_i,w_j)\, b(z_i,w_j)}\right)_{1\le i,j\le N}$$

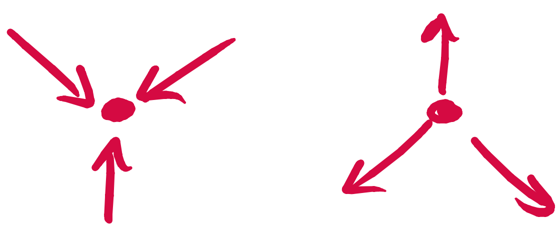
$V(z_i,w_j)$ [cf J. Lamers' talk]

2. TRIANGULAR ICE (20V model)





ice rule



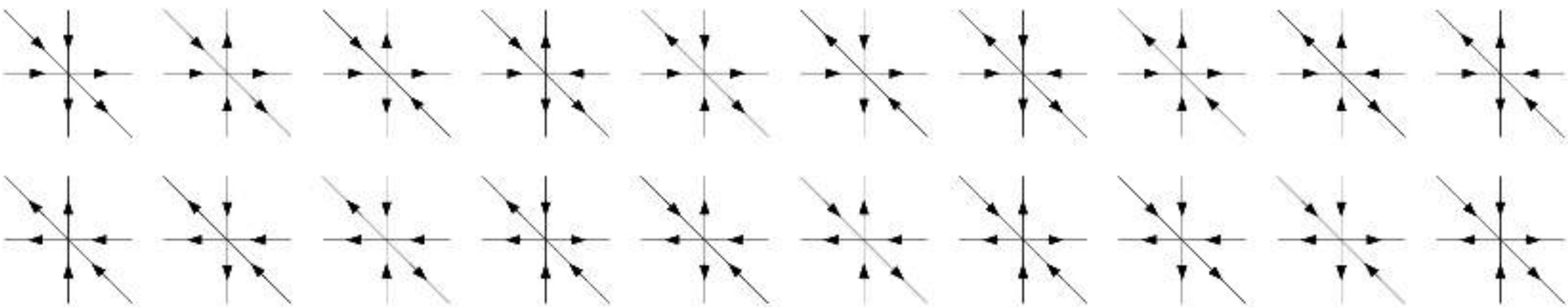
at each vertex

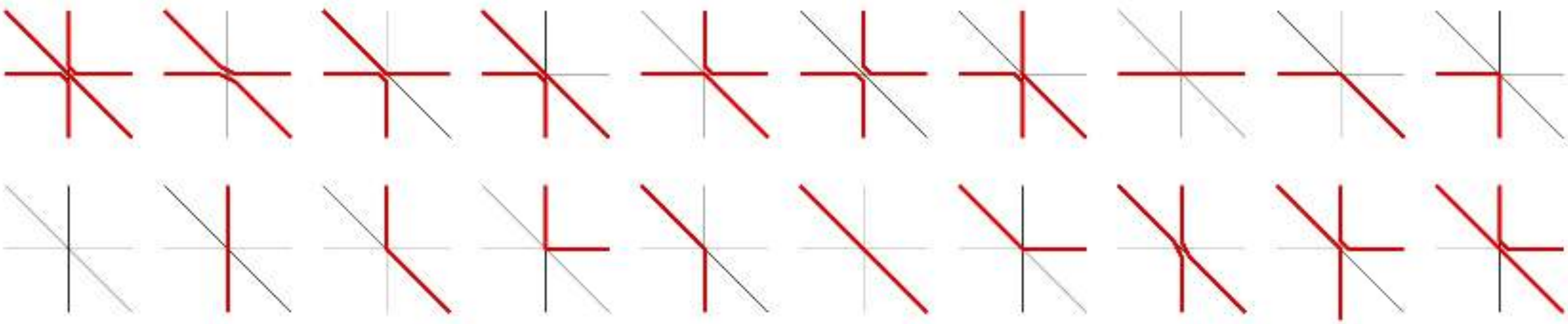
[Kelland, Baxter]

$$\binom{6}{3} = 20$$

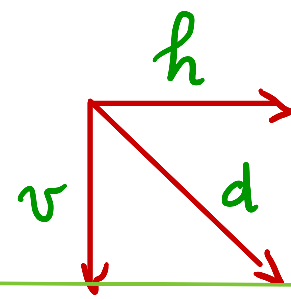
TRIANGULAR ICE (20V model)

Twenty vertices:





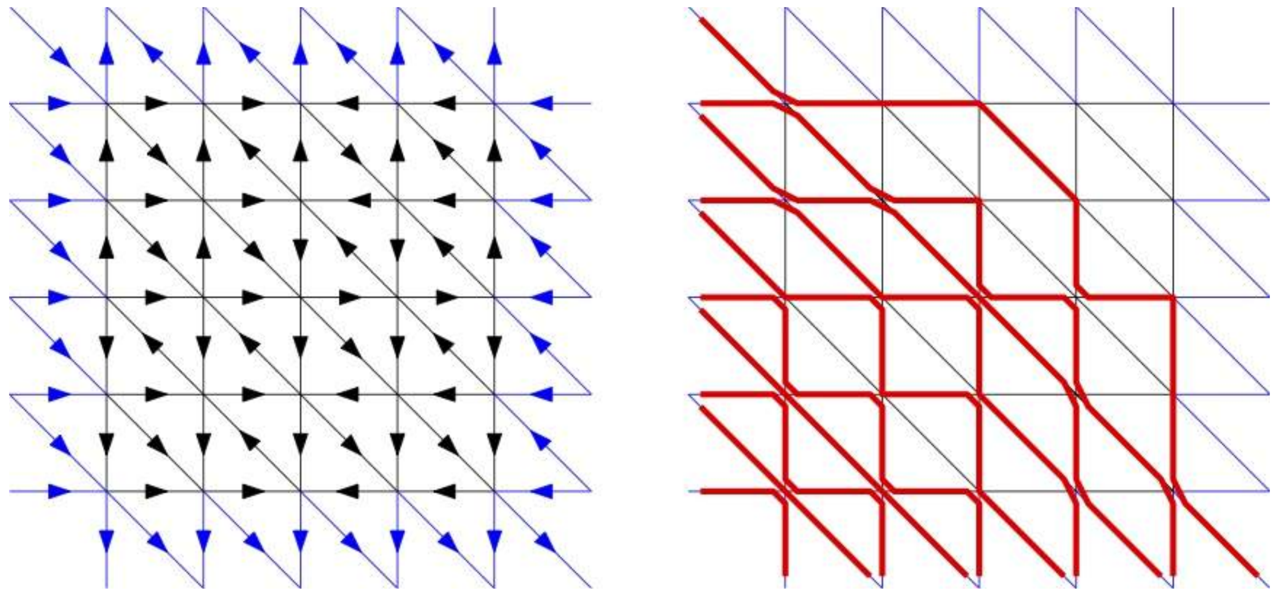
Osculating Schröder paths



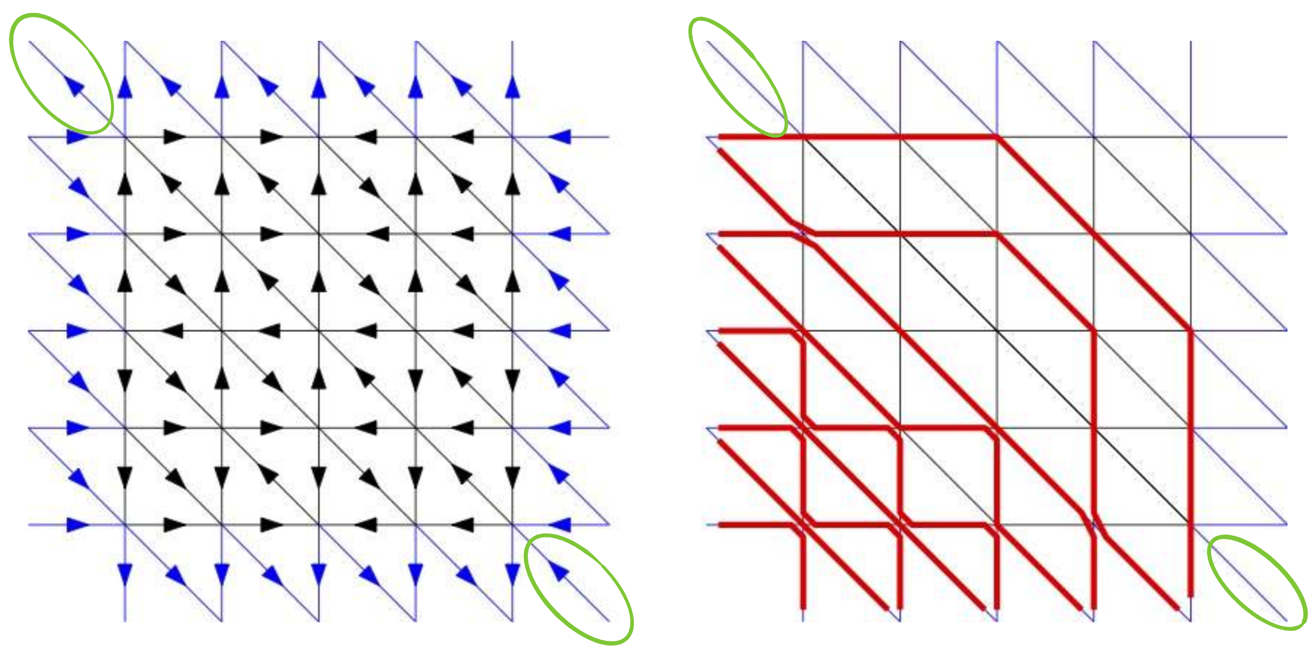
h, v, d steps

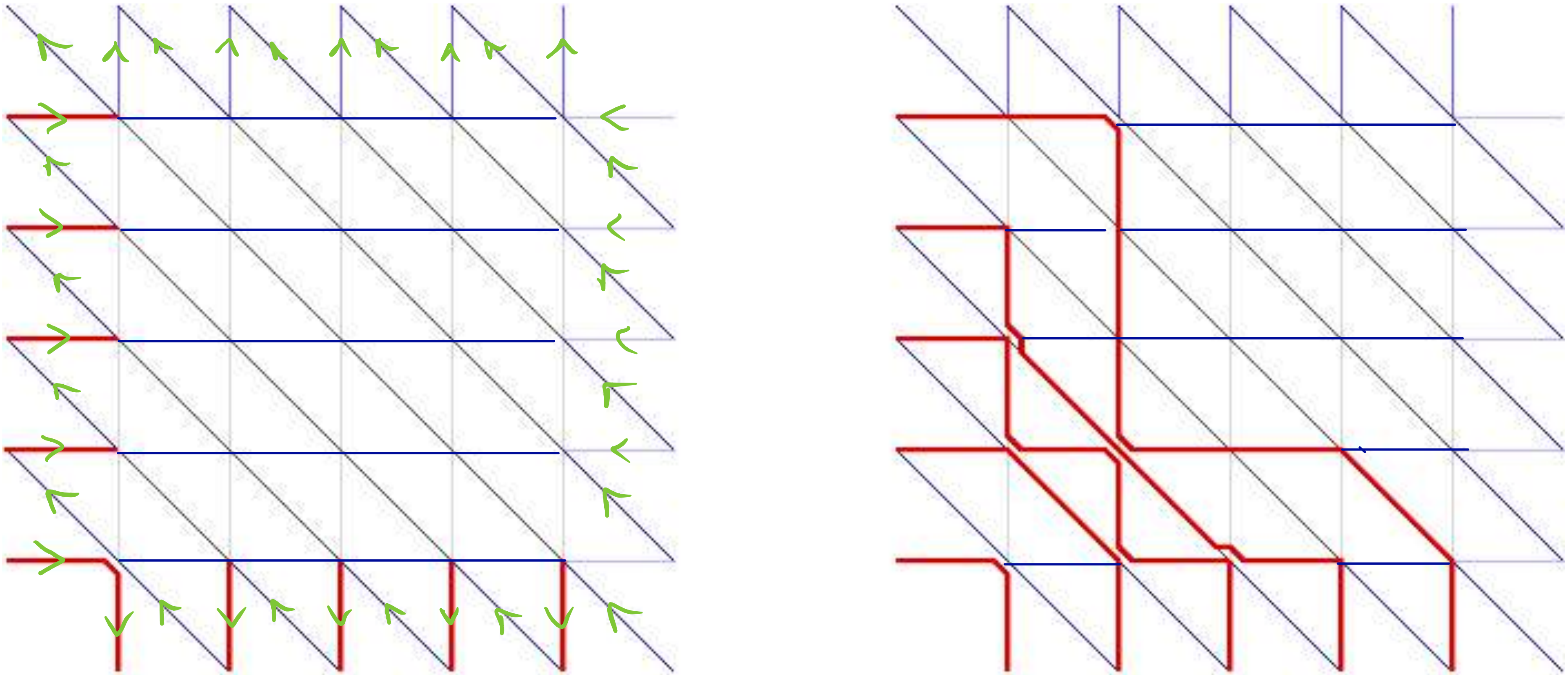
DOMAIN WALL BOUNDARY CONDITIONS

DWBC1



DWBC2





DWBC 3

Numbers of Configurations on an $n \times n$ grid:

DWBC 1,2

$$A_n = 1, 3, 23, 433, 19705, 2151843, \ldots$$

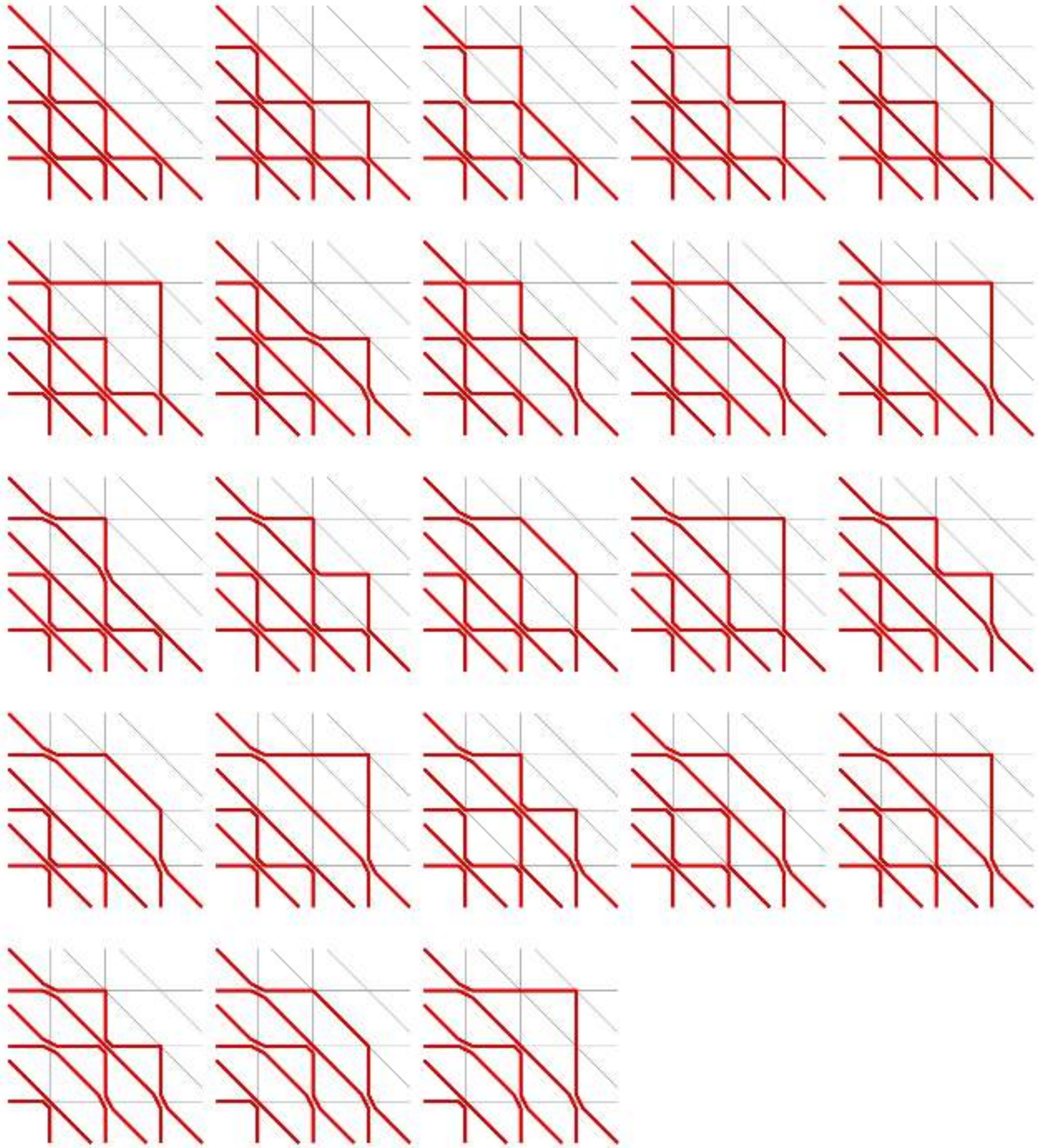
DWBC 3

$$B_n = 1, 3, 29, 901, 89893, 28793575, \ldots$$

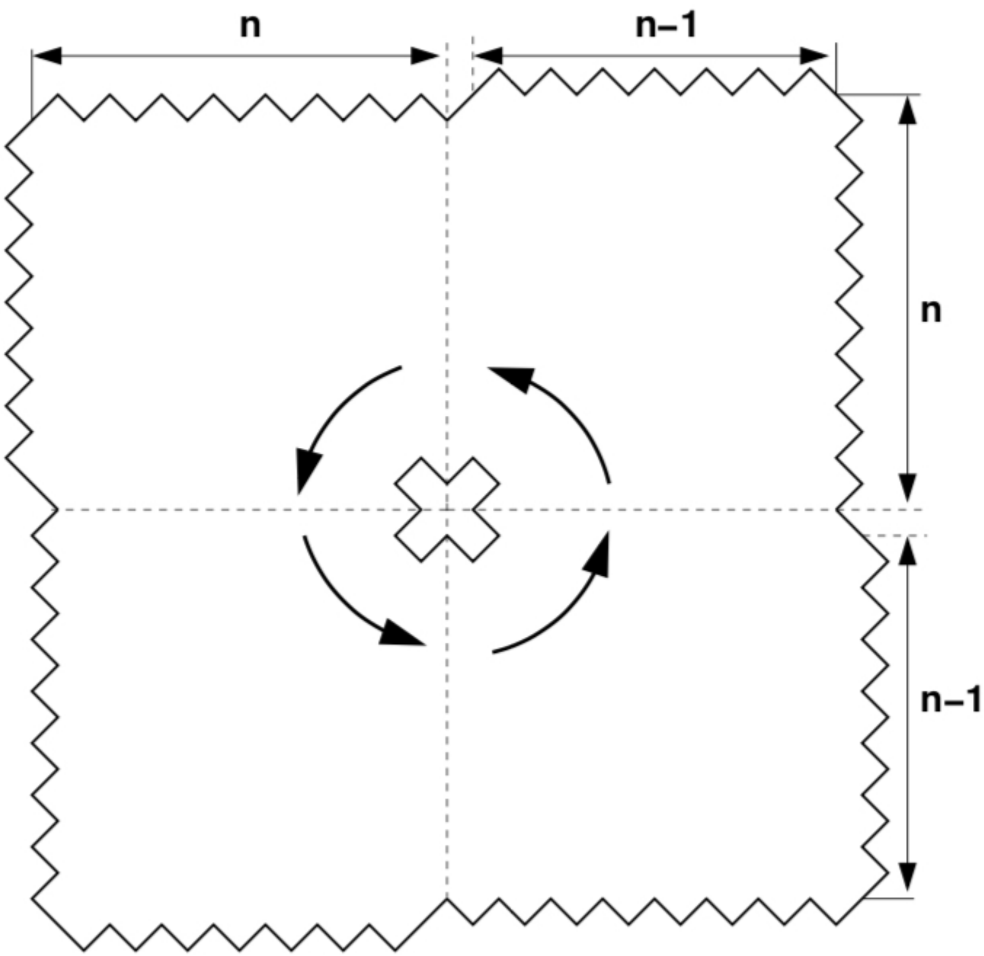
[computed by transfer matrix]

20V
DWBC1
configurations
$n = 3$

(23)



3. DOMINO TILINGS OF THE HOLEY AZTEC SQUARE WITH QUARTER-TURN SYMMETRY



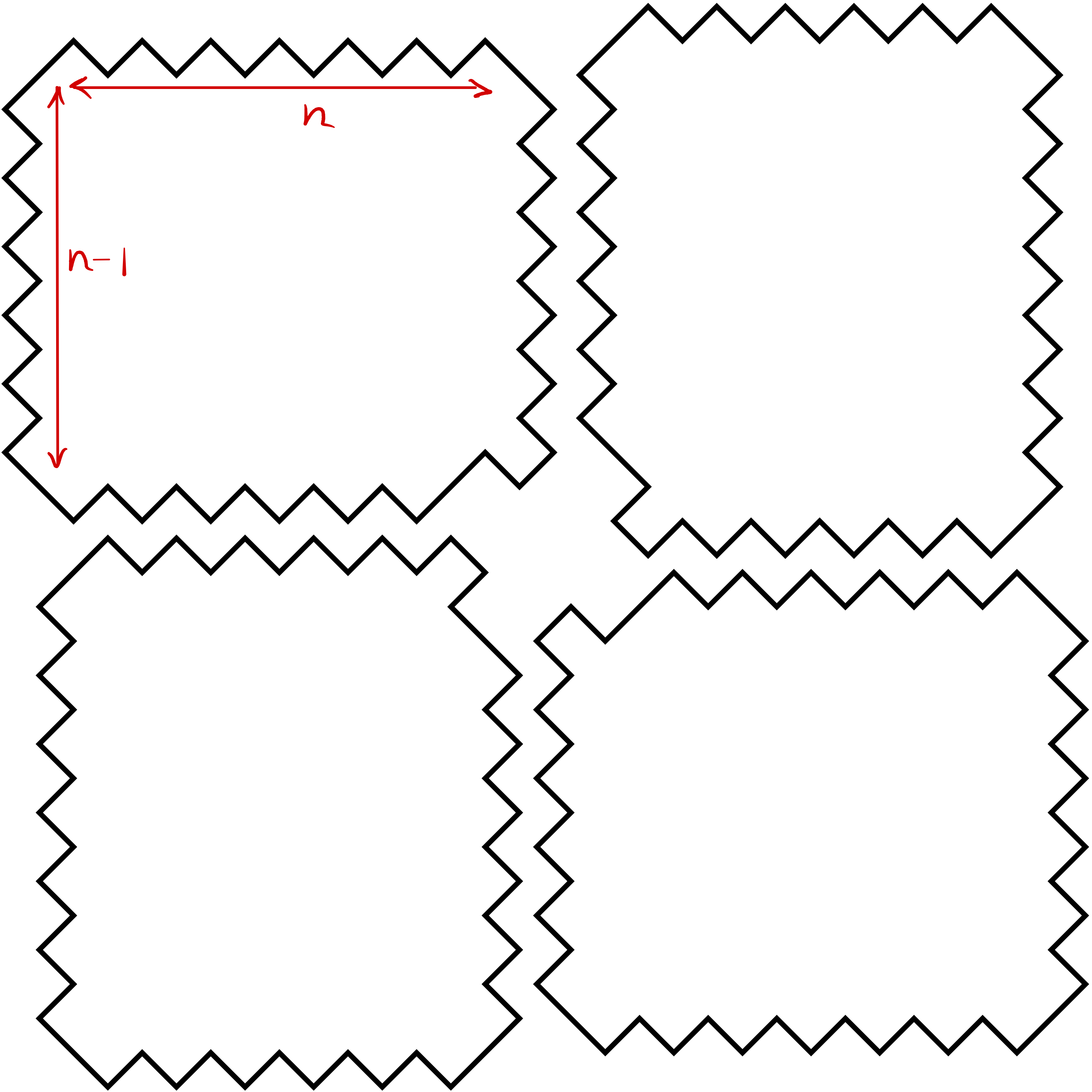
$\mathcal{S}_n$

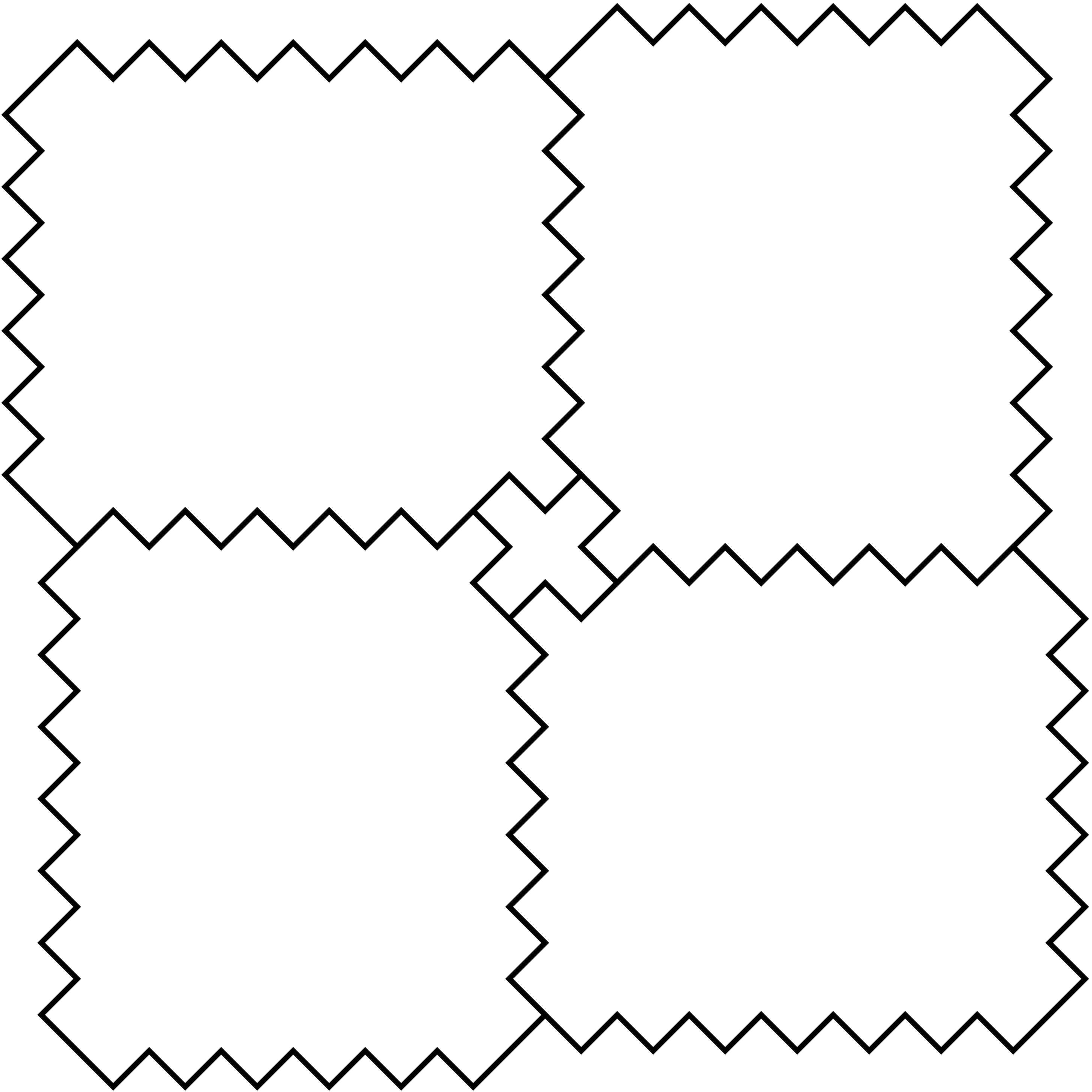
Domino Tilings: use

and 2×1 dominos

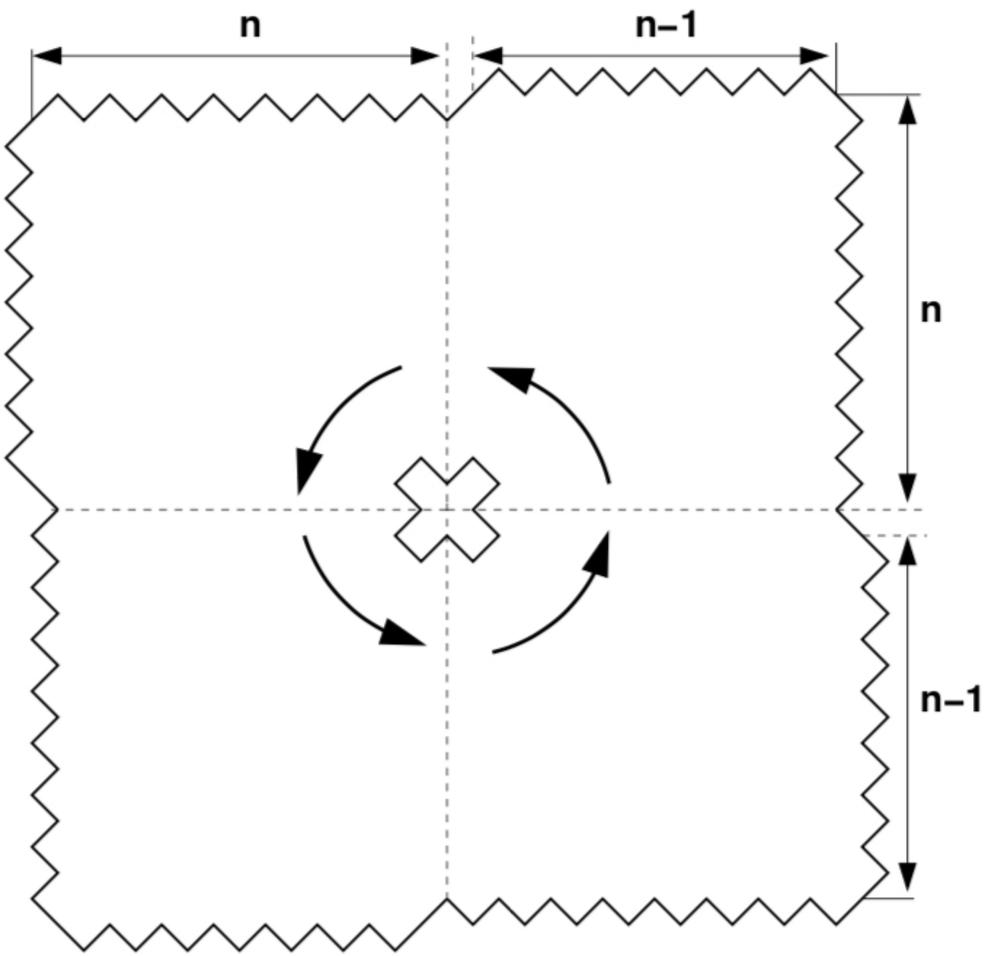
Rotational symmetry by $\frac{\pi}{2}$

NB: the hole makes it tileable!

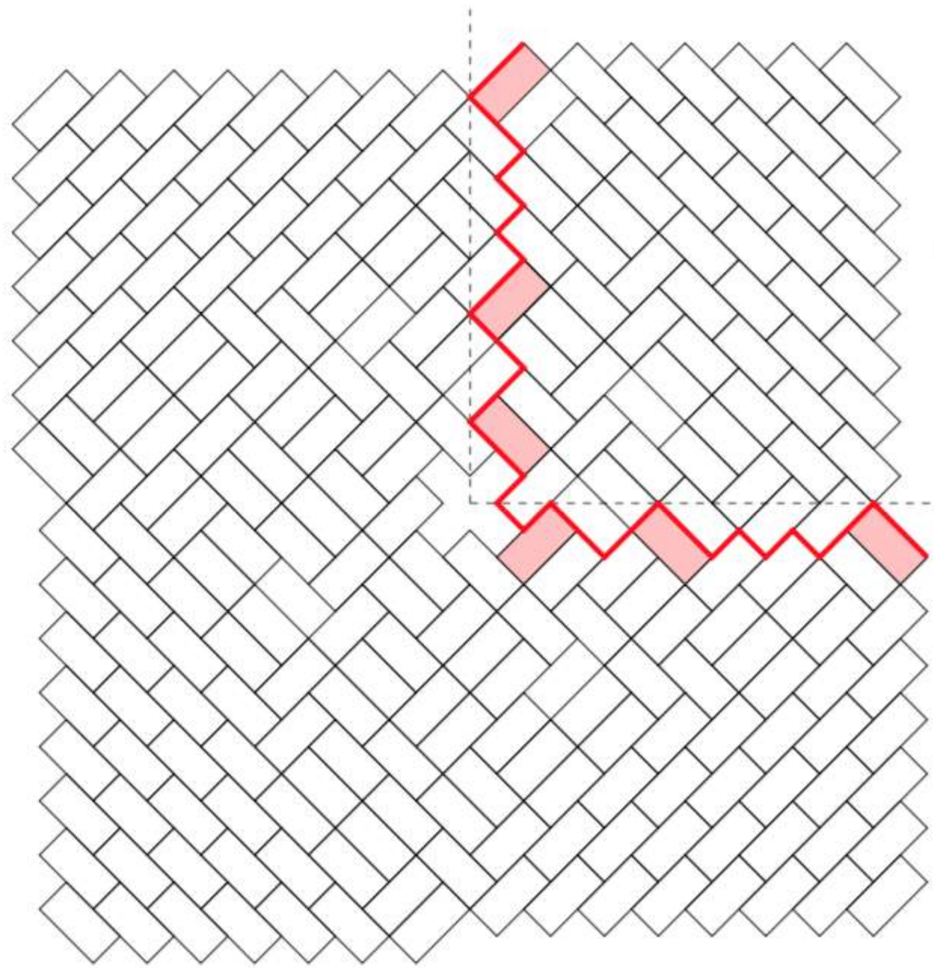

n
n-1



DOMINO TILINGS OF THE HOLEY SQUARE WITH QUARTER-TURN SYMMETRY

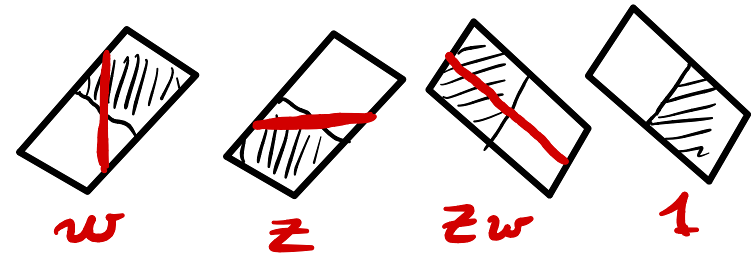


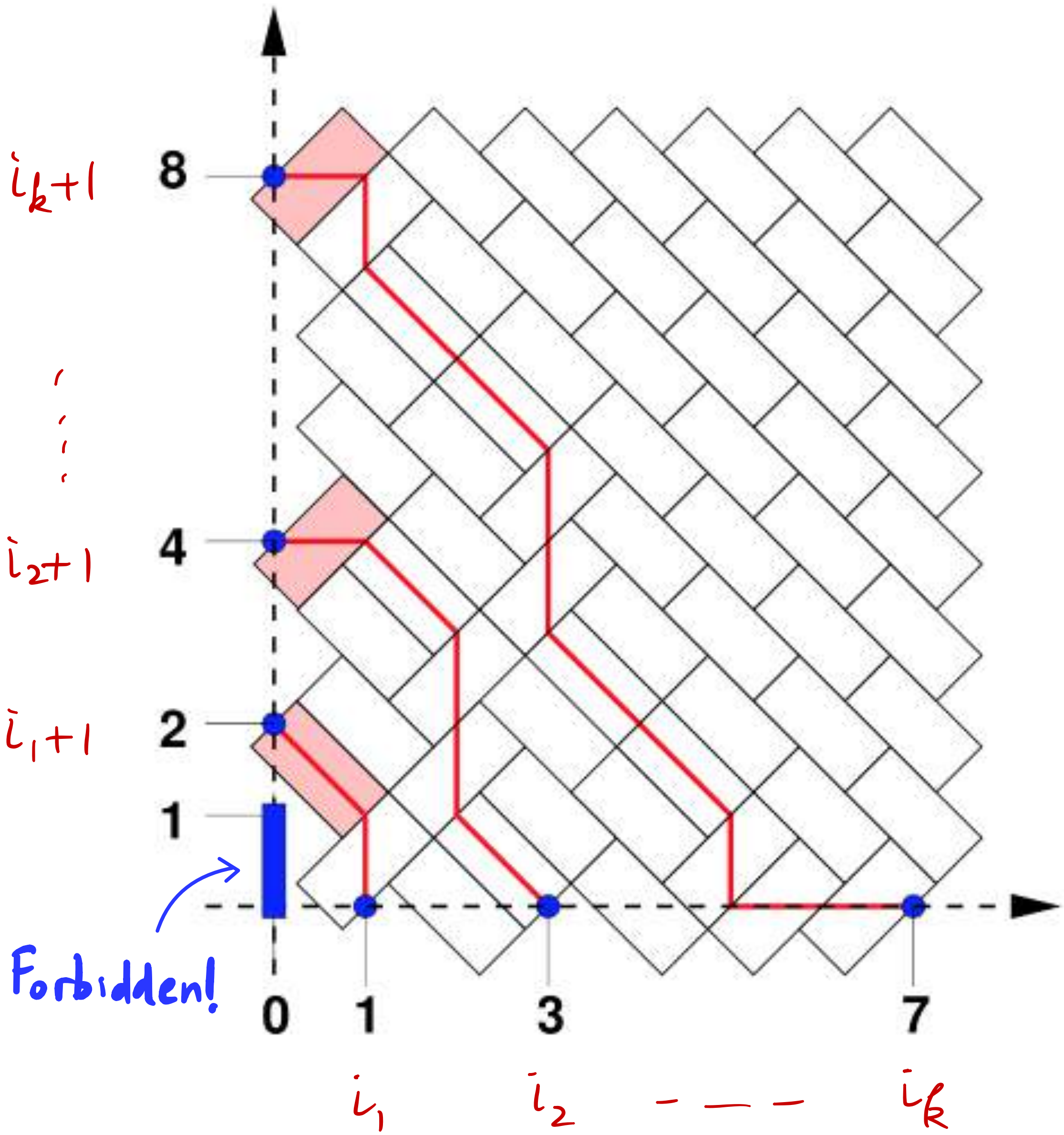
$\mathcal{S}_n$



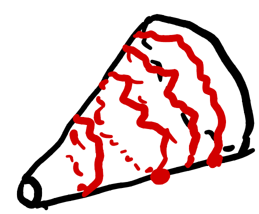
sample tiling

Counting Configurations





- Non-intersecting Schröder paths w fixed ends
- first step cannot be |
- start and ends identified (cone).



Counting Configurations

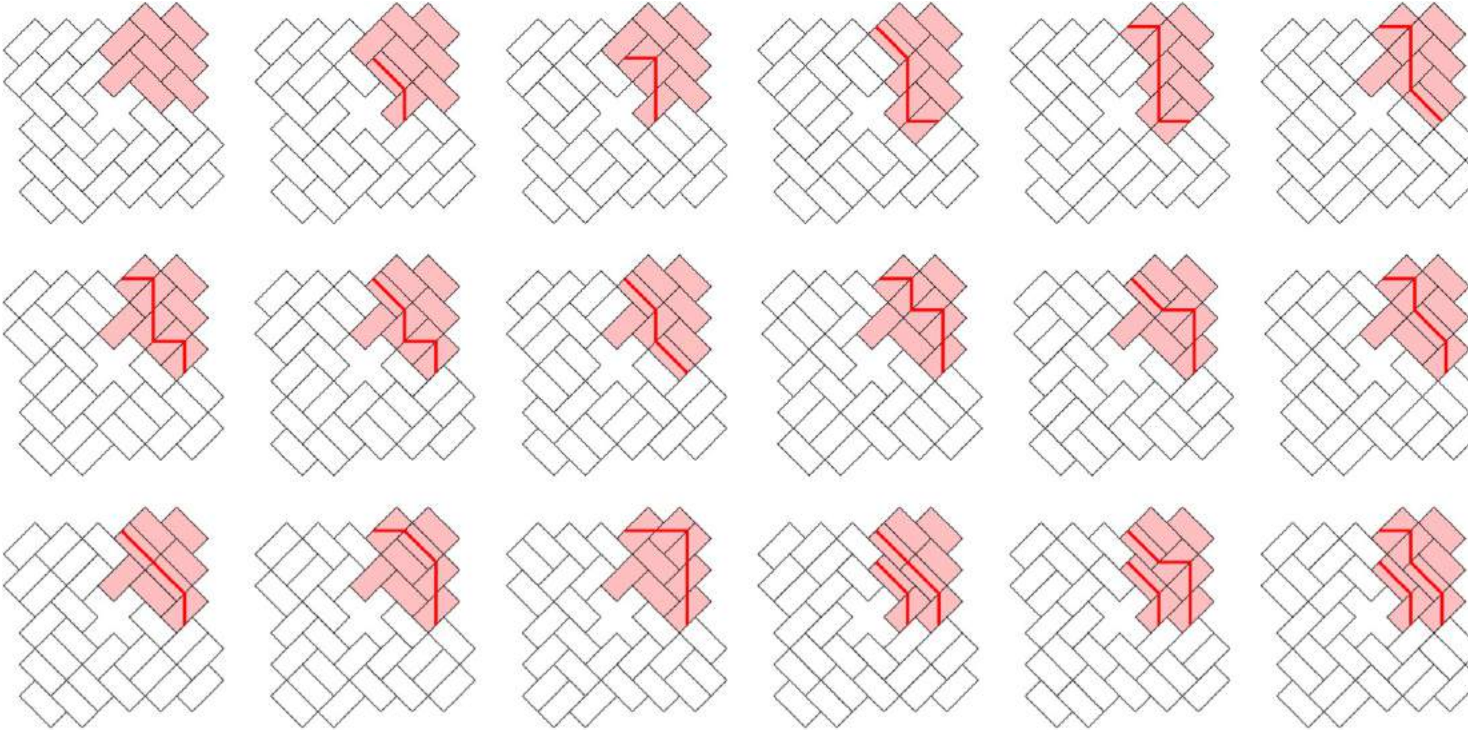
Thm [PDF-Guitter 19]

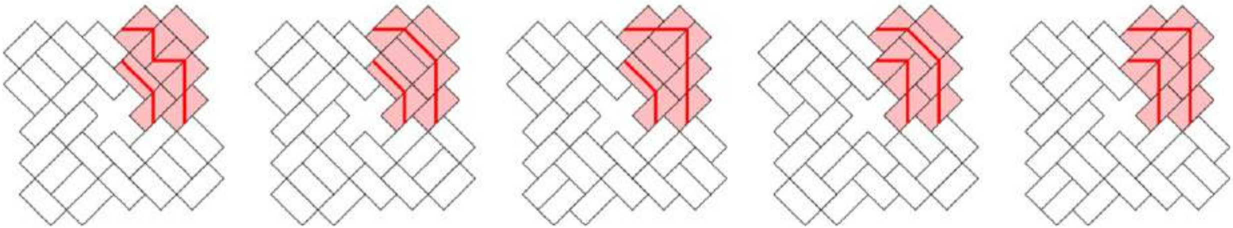
$$T_4(\mathcal{T}_n) = \det_{0\le i,j\le n-1}\left(\left\{\frac{1}{1-zw} + \frac{2z}{(1-z)(1-z-w-zw)}\right\}_{z^i w^j}\right)$$

Proof: (Cauchy Binet) $\det(Id + M) = \sum_{i_1<\dots<i_k} \underbrace{\left|M^{i_1\dots i_k}_{i_1\dots i_k}\right|}_{\text{(Gessel-Viennot)}}$

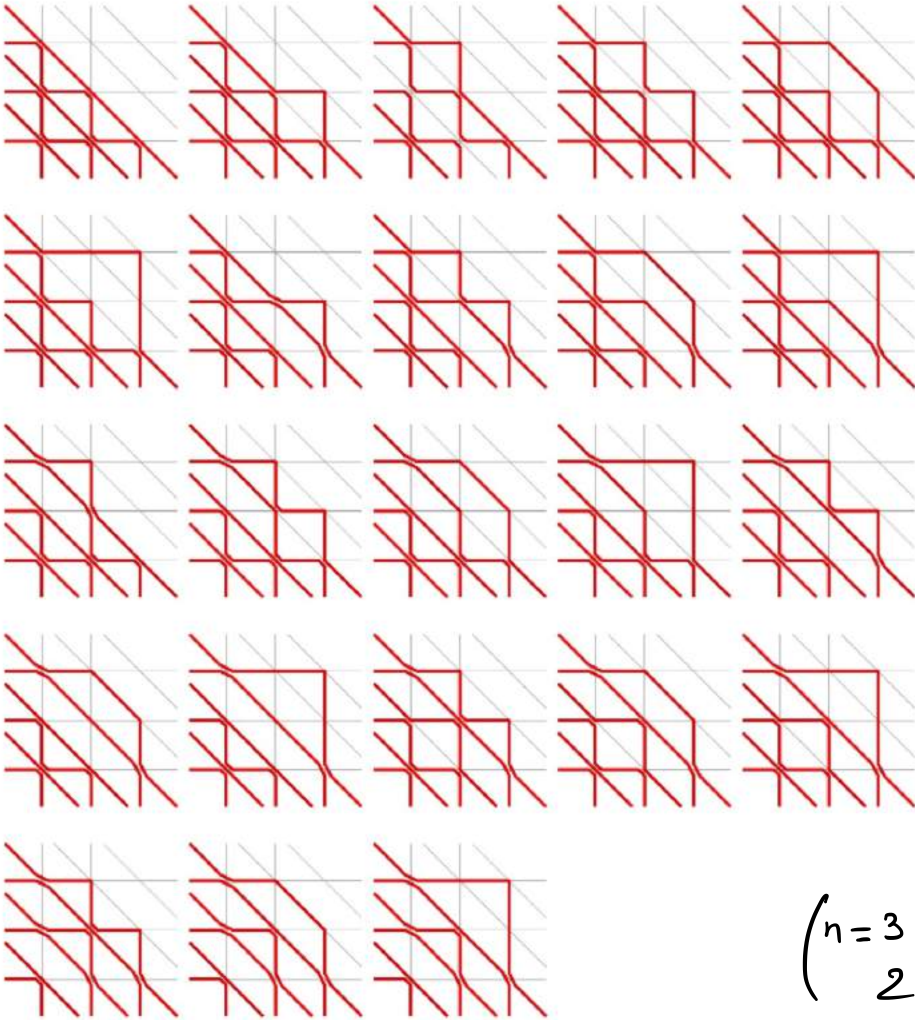
$$T_4(\mathcal{T}_n) = 1, 3, 23, 433, 19705, 2151843, \dots$$

Ex: $n=3$ $\det\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\2&2&2\\4&8&12\end{pmatrix}\right] = 23$ Domino Tiling configurations $\rightarrow$





$\left(\begin{array}{c} n=3 \\ \text{holey square} \\ \text{Domino Tilings} \end{array} \right)$



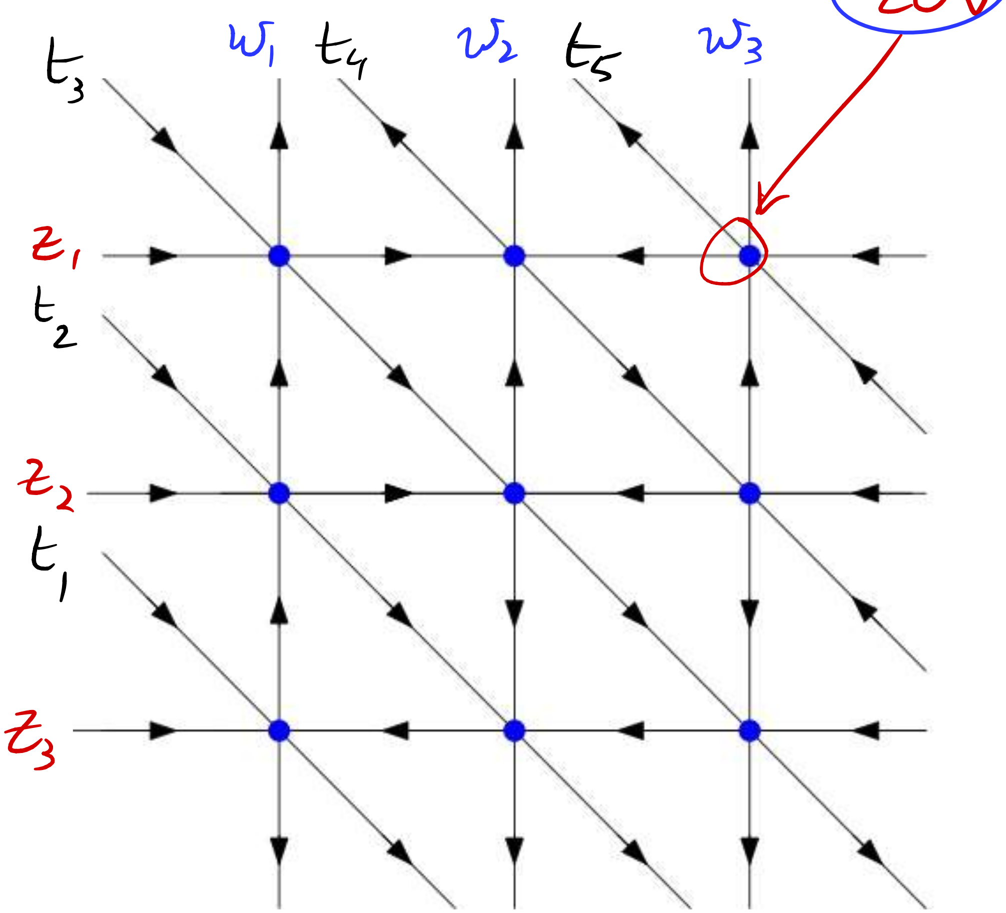
(n = 3 DWBC1
20V configurations)

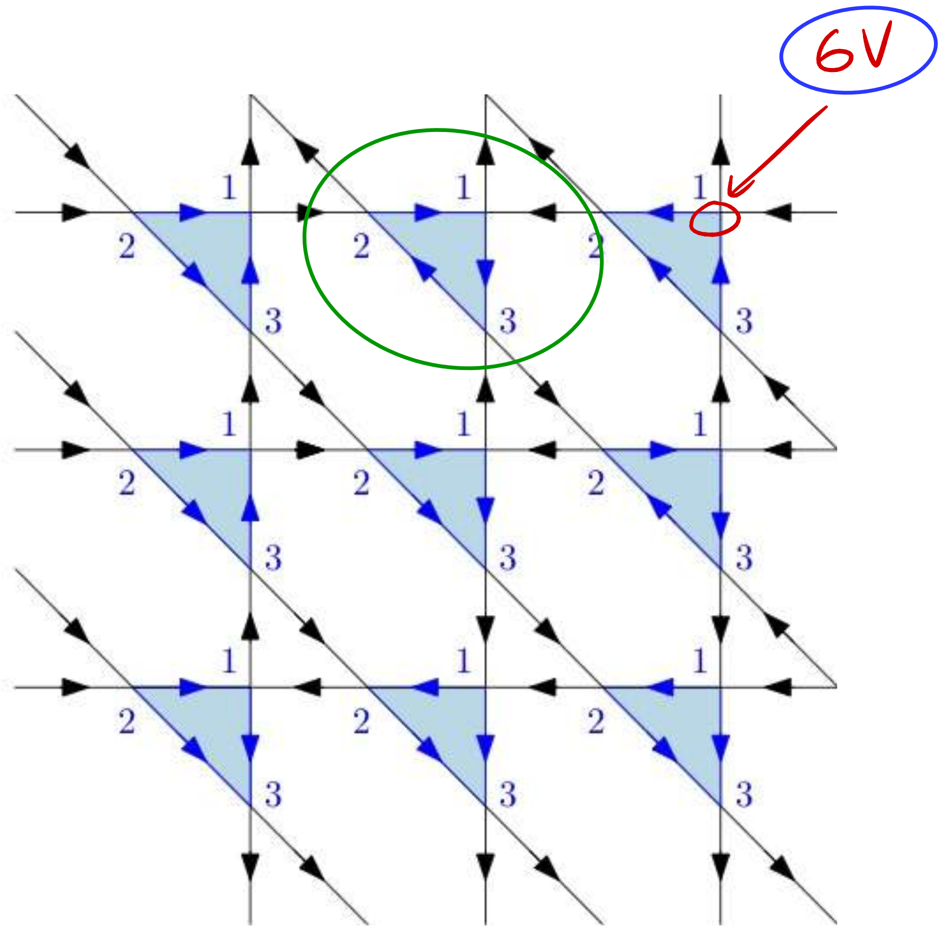
4. PROOF OF THE CORRESPONDENCE WITH 20V - DWBC1,2

idea

- use integrable weights for the 20V
- deform the line arrangement into a 6V
- use 6V results (Izergin-Korepin det)
- refinement

ICE MODEL ON THE KAGOME LATTICE



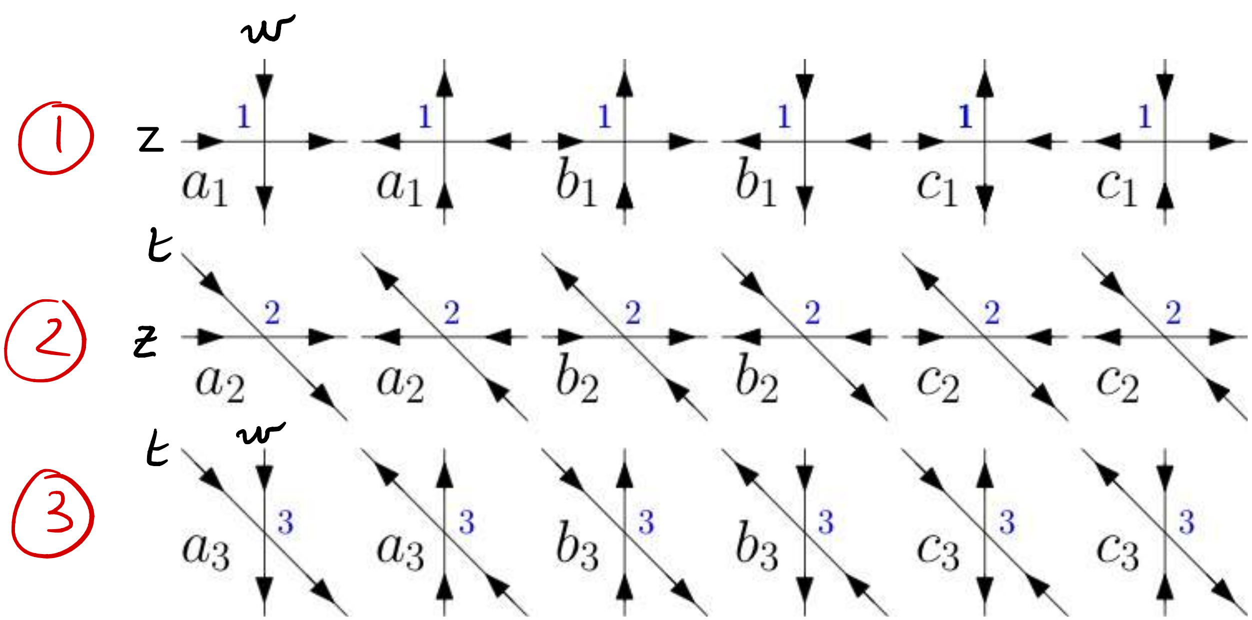


Triangular lattice ice

Kagome Lattice ice

20V weights ⟵ 6V weights on 3 sublattices 1,2,3

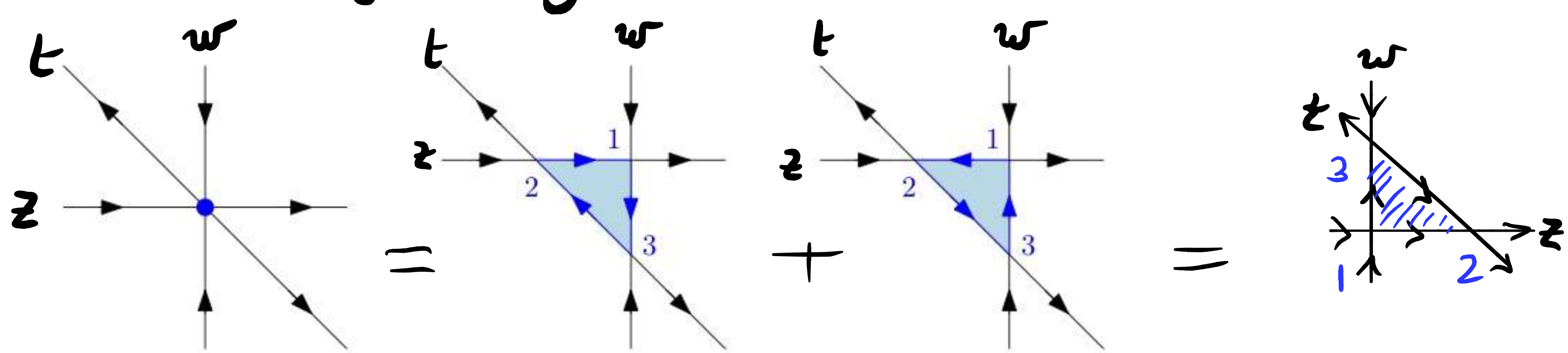
BOLTZMANN WEIGHTS



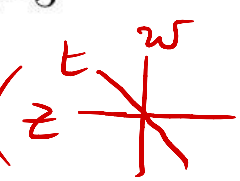
weights of the 6V models on the 3 sublattices

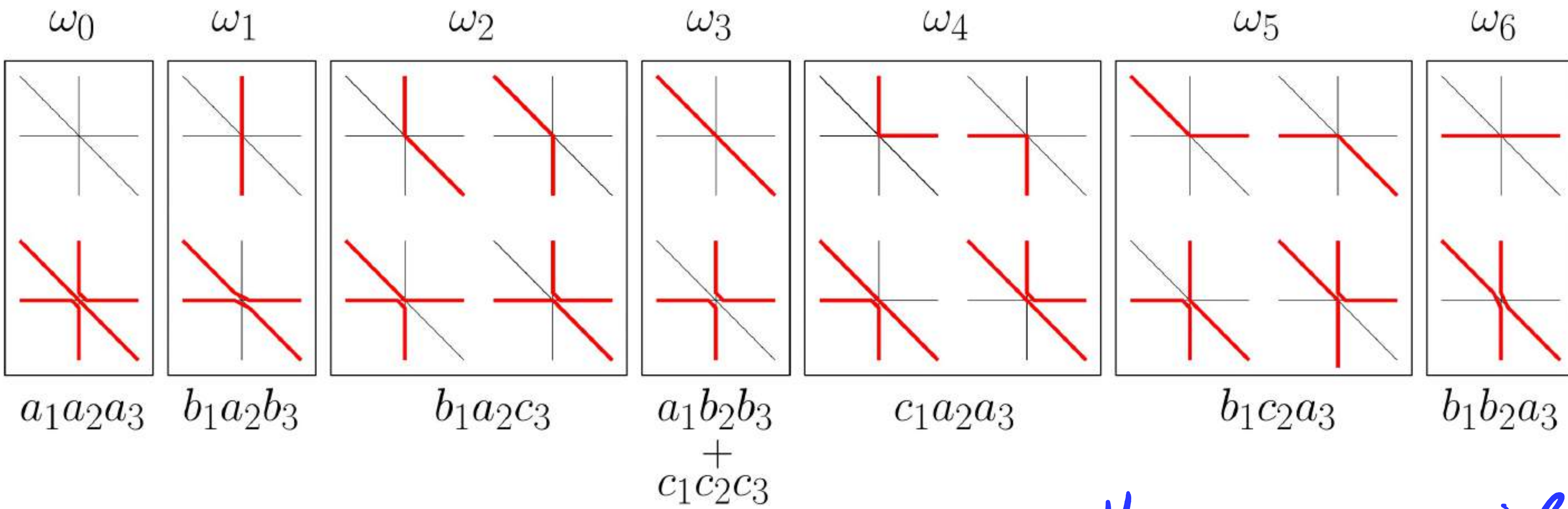
- 20V weights are given by sums over inner triangle configs

Example:



$$W = a_1 b_2 c_3 + c_1 c_2 b_3 = b_1 a_2 c_3$$

- Homogeneous case: 3 parameter family: ( , q)



$$\omega_0 = \sin(\lambda+\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_1 = \sin(\lambda-\eta)\sin\left(\frac{\lambda-\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_2 = \sin(2\eta)\sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_3 = \sin(2\eta)^3 + \sin(\lambda+\eta)\sin\left(\frac{\lambda-\eta+\mu}{2}\right)\sin\left(\frac{\lambda-\eta-\mu}{2}\right)$$

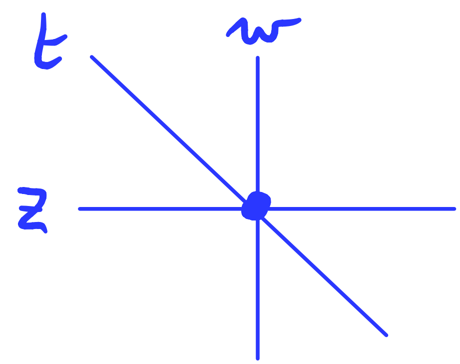
$$\omega_4 = \sin(2\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_5 = \sin(2\eta)\sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)$$

$$\omega_6 = \sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda-\eta-\mu}{2}\right),$$

Homogeneous weights:

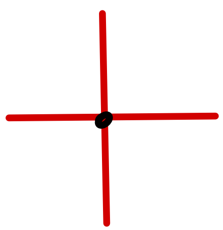
$$\begin{cases} q = e^{i\eta} \\ z = e^{i(\eta+\lambda)} \\ w = e^{-i(\eta+\lambda)} \\ t = e^{i\mu} \end{cases}$$

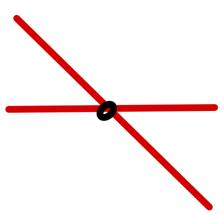


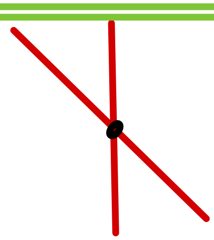
Remark: uniform weights $\omega_i = 1 \;\; \forall i$ are obtained for:

$$\boxed{\eta = \frac{\pi}{8} \qquad \lambda = \frac{5\pi}{8} \qquad \mu = 0}$$

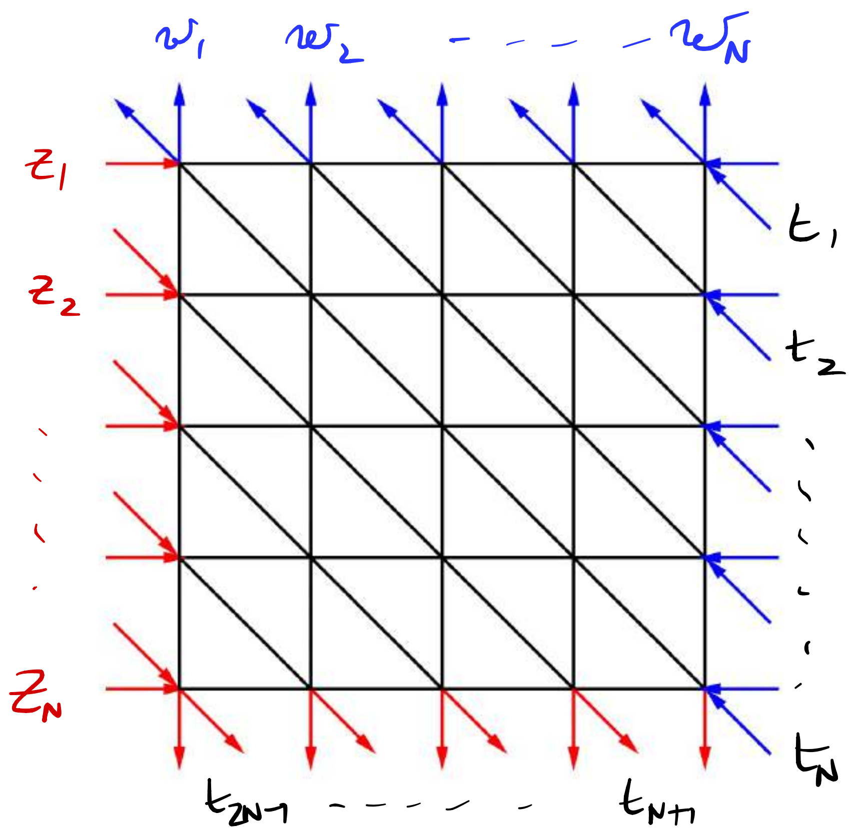
This corresponds to non-uniform weights on the 3 sublattice 6V models (up to an overall factor).

1  $a_1 = 1 \qquad b_1 = \sqrt{2} \qquad c_1 = 1$

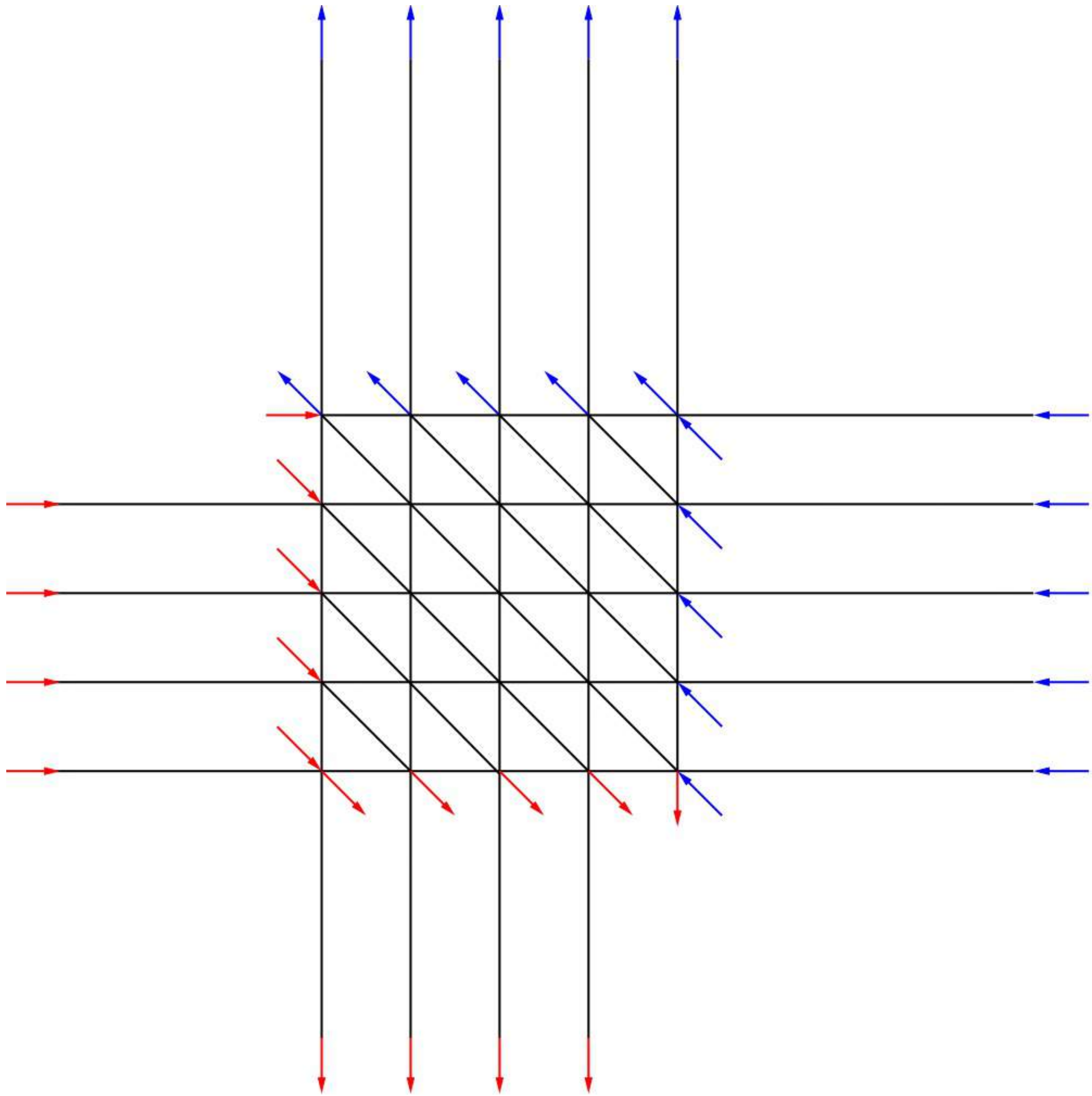
2  $a_2 = \sqrt{2} \qquad b_2 = 1 \qquad c_2 = 1$

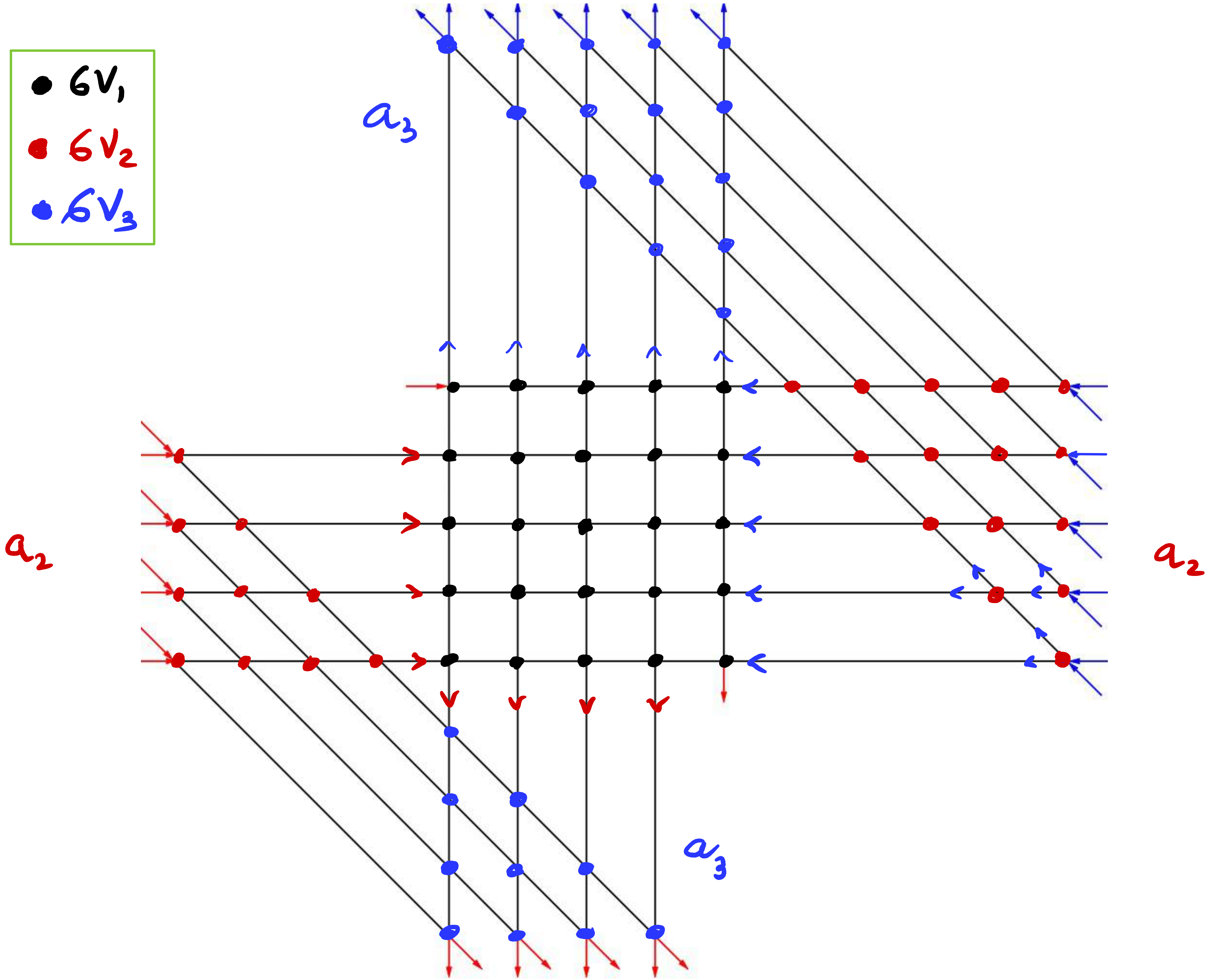
3  $a_3 = \sqrt{2} \qquad b_3 = 1 \qquad c_3 = 1$

TRANSFORMATION INTO a 6V MODEL

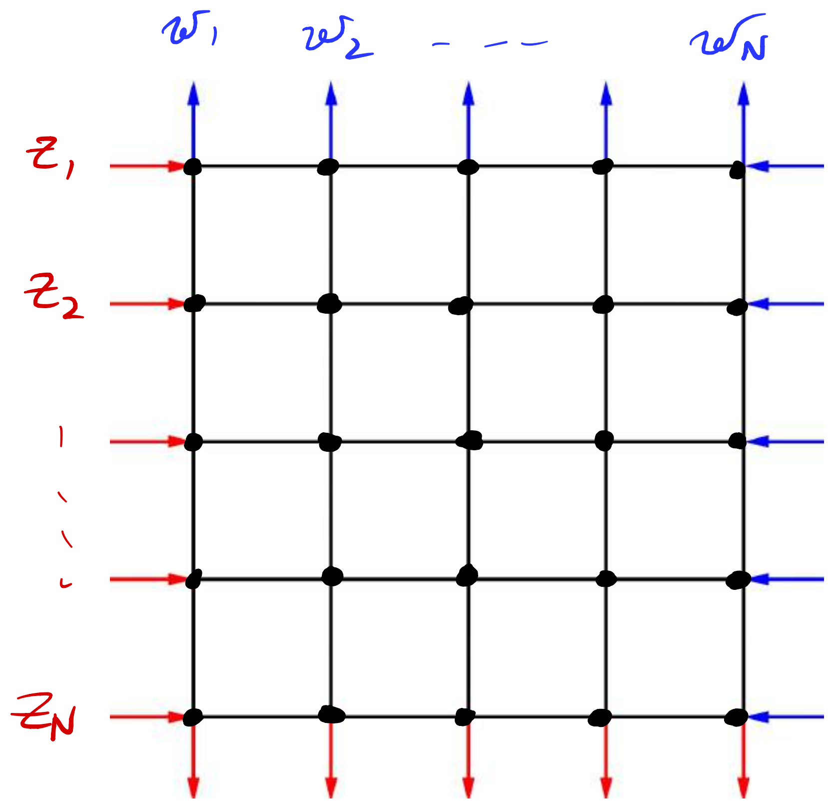


20V DWBC-2 (integrable weights).



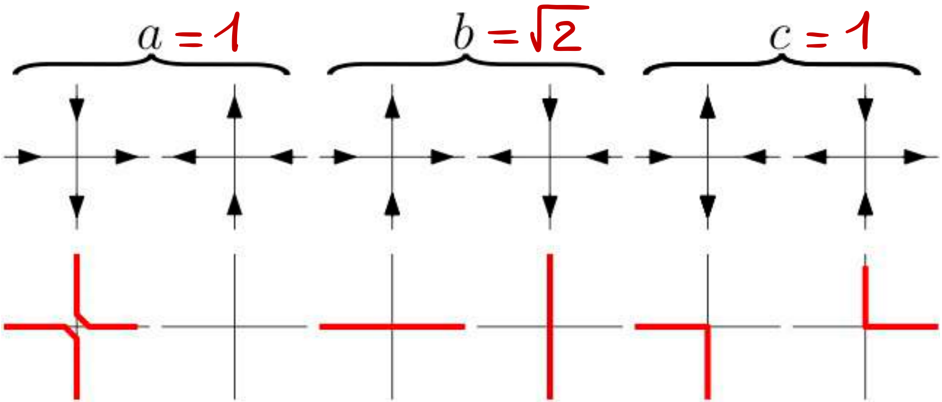


$6V_1$
$6V_2$
$6V_3$
a_3
a_2
a_2
a_3



6V DWBC (sublattice 1 only).

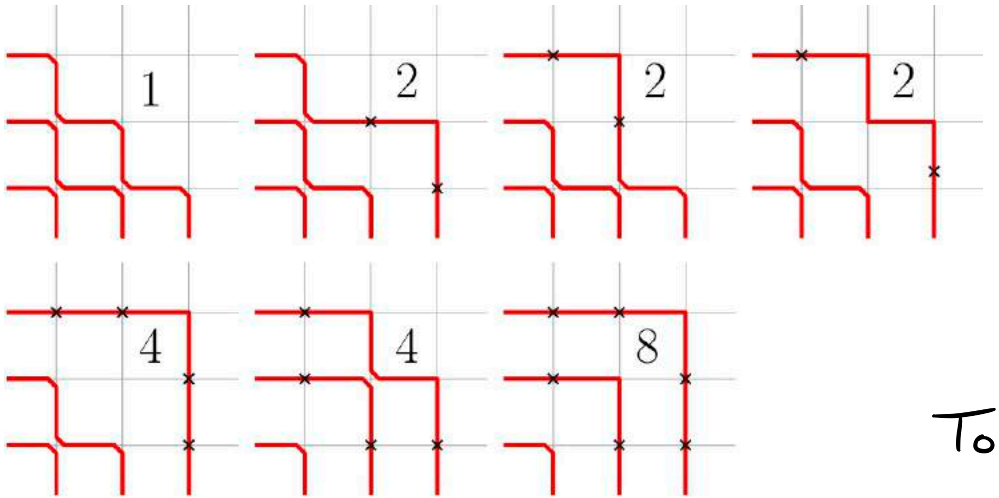
Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights = 1 is equal to that of the 6V model with weights $(a,b,c) = (1, \sqrt{2}, 1)$ and DWBC

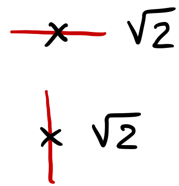


A lattice of KAGOME (Daikokuya, Kitashirakawa)

Example of size $n=3$

20V-DWBC1 vs 6V aka ASM





(b weights)

Total $=$ 23 $\mathbf{APM}_1$ of size 3

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\left.\begin{matrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\end{matrix}\right\} \text{7 ASM of size 3.}$$

Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights $=1$ is equal to that of the 6V model with weights $(a,b,c)=(1,\sqrt{2},1)$ and DWBC

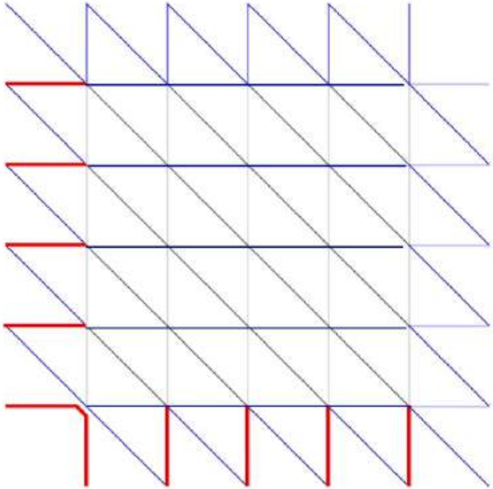
Then use classical result by Korepin-Izergin for the 6V-DWBC and spectral parameters $(z_1 \ldots z_n, w_1 \ldots w_n)$

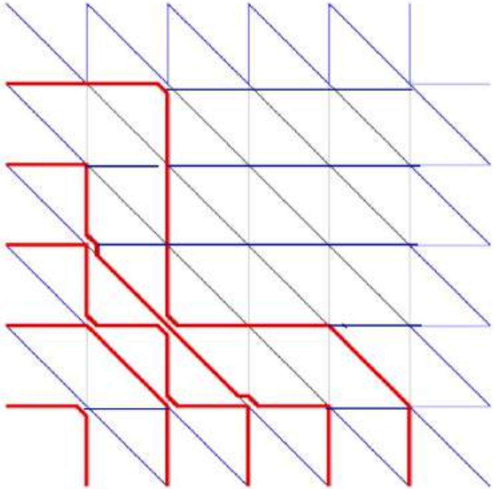
$$Z_{6VDWBC}(z_1 \ldots z_n, w_1 \ldots w_n) = \frac{\prod_{i=1}^{n} c(z_i, w_i) \prod_{i,j=1}^{n} a(z_i, w_j)\, b(z_i, w_j)}{\prod_{1 \le i < j \le n} (z_i - z_j)(w_i - w_j)} \det_{1 \le i,j \le n} \left\{ \frac{1}{a(z_i, w_j)\, b(z_i, w_j)} \right\}$$

→ Limiting procedure → same det as holey square DT !
(cf Behrend, PDF, Zinn Justin)

→ Refinements

5. The DWBC 3 Conjectures





DWBC 3

$b_n = 1, 3, 29, 901, \ldots$

The DWBC 3 Conjectures

[OEIS for B_n] $\longrightarrow$ Domino Tilings of a $2n \times 2n$ square

$$= 2^n b_n^2$$

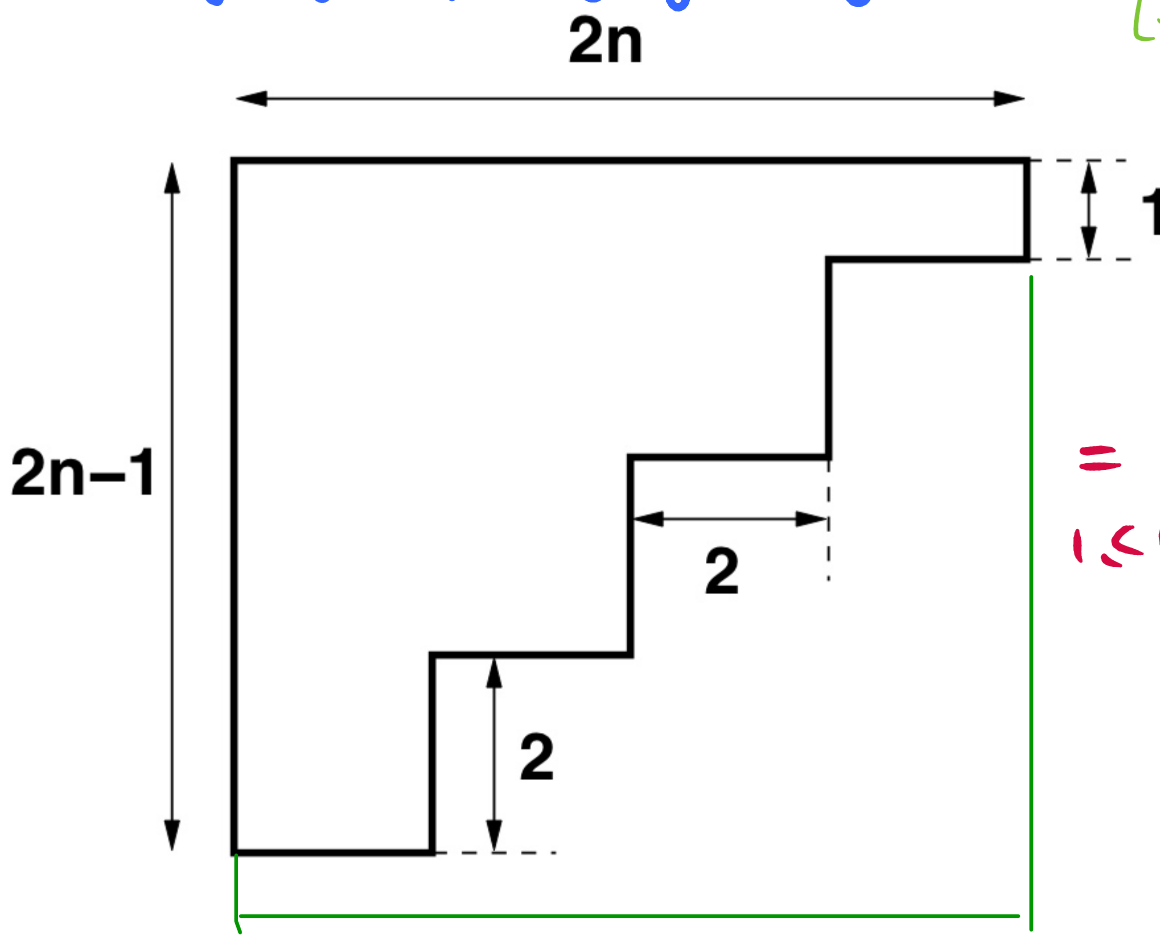
$b_n = 1, 3, 29, 901, \ldots$

[Temperley–Fisher 61]

[Pachter 97] proof of integrality of b_n: found a Domino Tiling interpretation

[see also Ciucu]

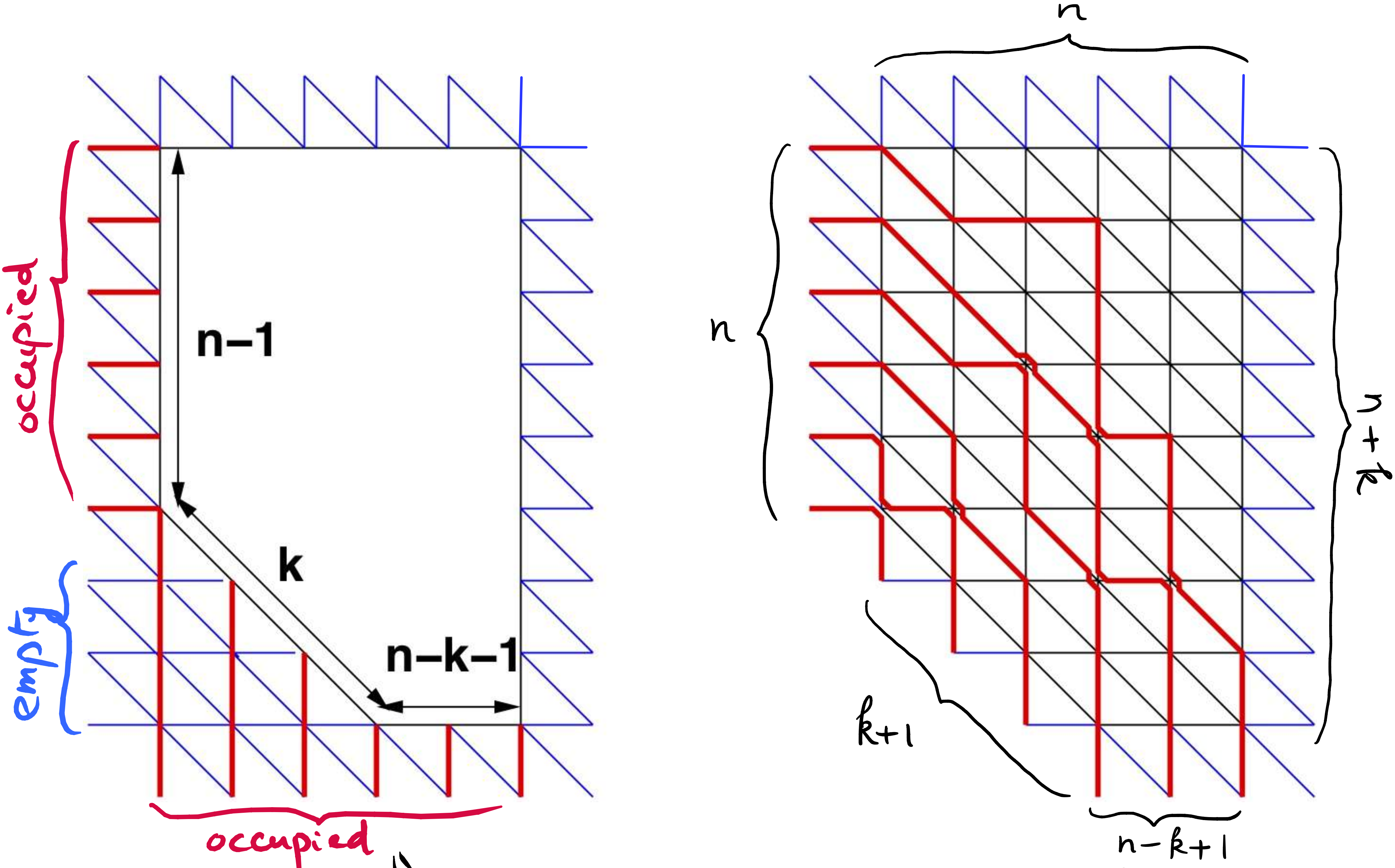
$b_n =$



$$= \prod_{1 \le i < j \le n} \left(4\cos^2 \frac{\pi i}{2n+1} + 4\cos^2 \frac{\pi j}{2n+1} \right)$$

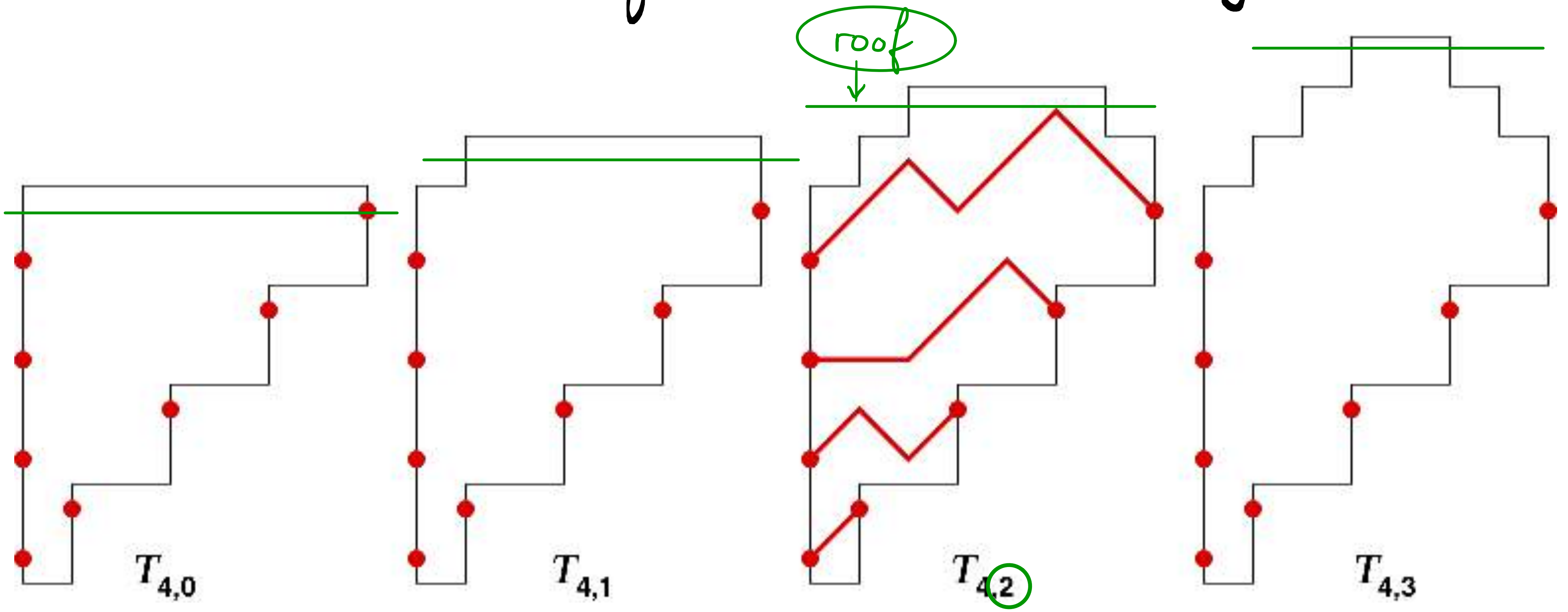
Conjecture 1. [PDF - E. Quitter 19] The configurations of the 20V-DWBC3 model on an $n \times n$ grid are counted by the Domino Tilings of Pachter's triangle

But we can do better....

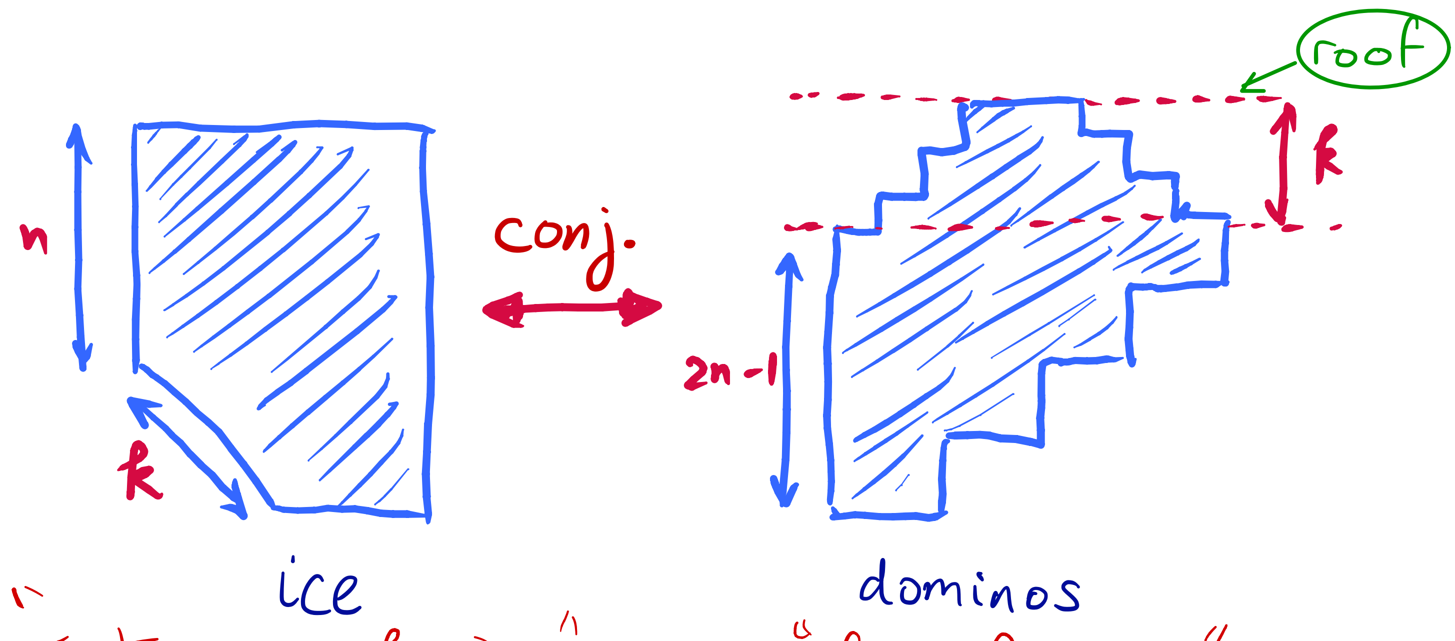


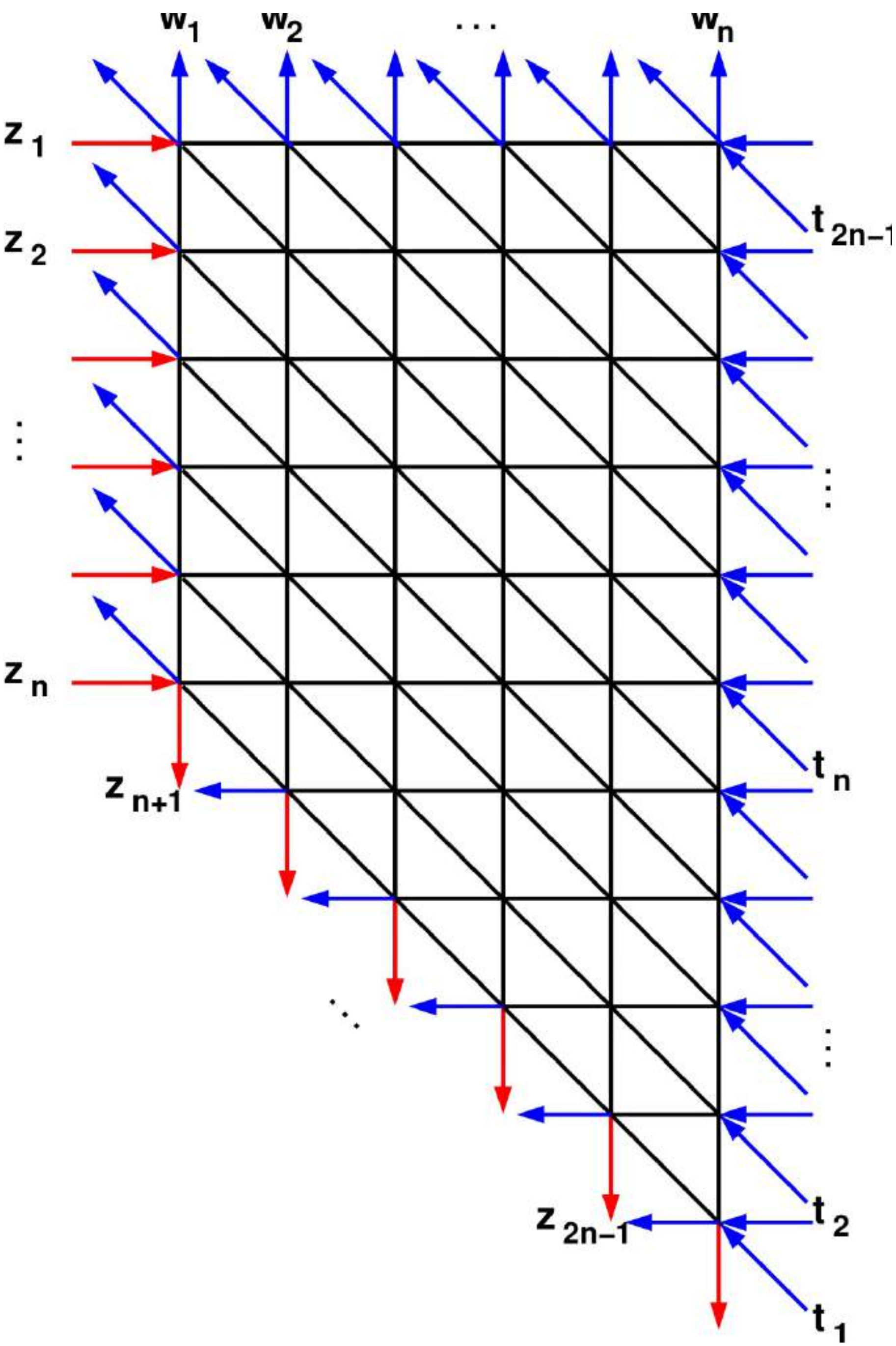
"Pentagon of triangular ice"

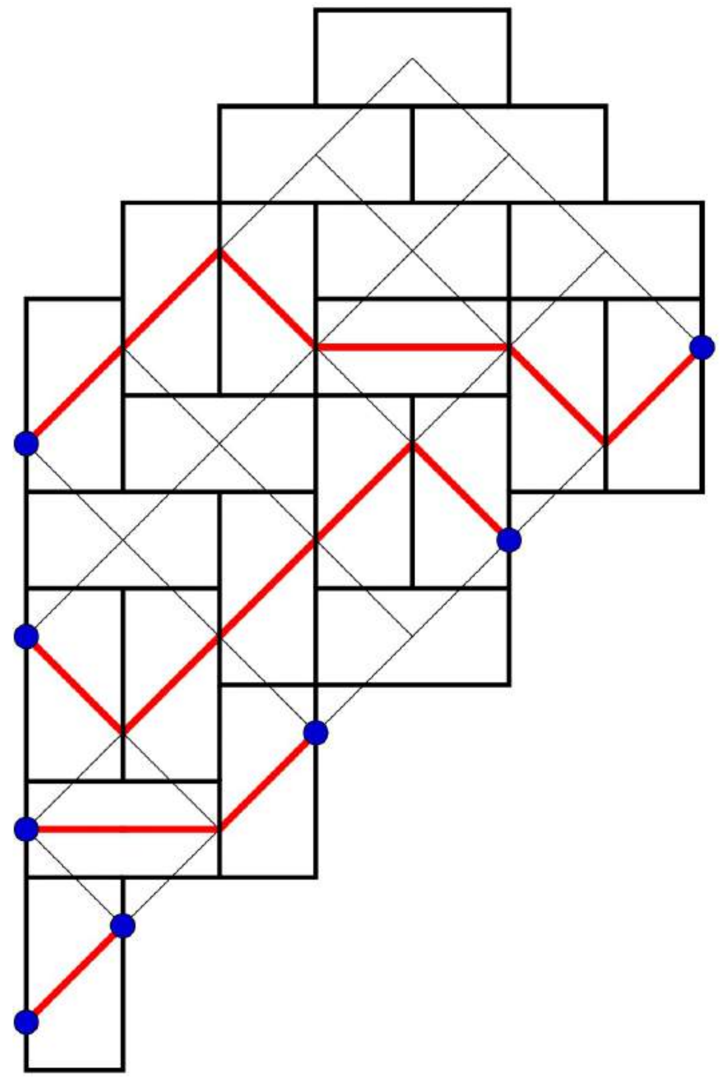
"raise the roof" above Pascher's triangle



Conjecture 2. [PDF+E. Guitter 19] The number of configurations of triangular ice in a pentagon w/ DWBC3 is equal to that of domino tilings of Pachter's raised triangle







DT of AztecTriangle$_n$

20V-DWBC 3 on Q_n

$$\boxed{Z_n = 1, 4, 60, 3328, \ldots}$$

Thm [Di Francesco 2021]

$$\boxed{Z^{20V\text{-}DWBC3}_{Q_n} = Z^{DT}_{AT_n}} = \det_{0\le i,j\le n-1}\left(\frac{1+u}{(1-uv)^2-v(1+u)^2}\bigg|_{u^iv^j}\right)$$

Conjecture 3

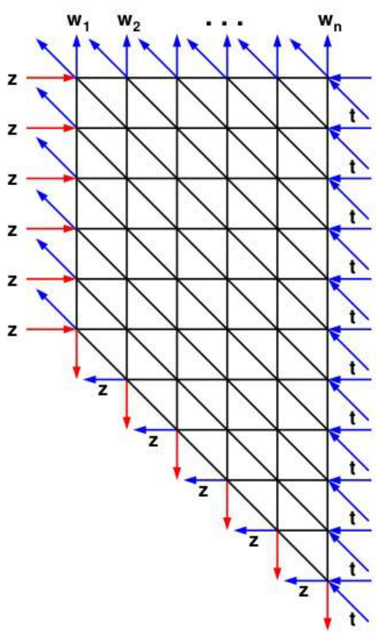
$$\boxed{Z^{20V\text{-}DWBC3}_{Q_n} = 2^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}}$$

$= 1, 4, 60, 3328, 678912, \ldots$

+ refinements

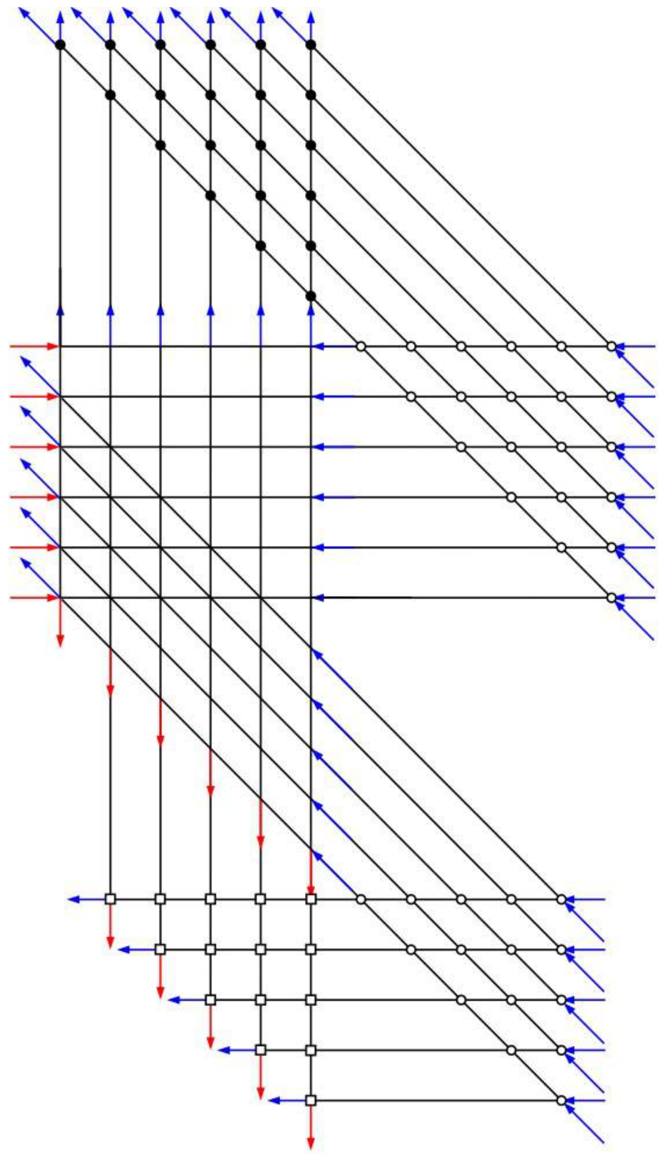
Progress [CIUCU '21] [Krattenthaler '21] [Trilai + PDF '21]

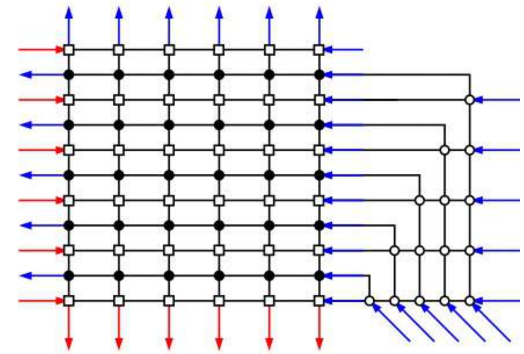
• Proof of Thm Along the same lines



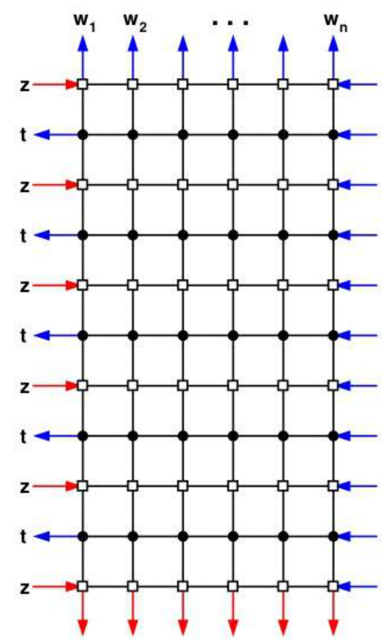
(a)

20V-DWBC3 on Q_n





(c)



6V with U-turn Boundary

[Kuperberg, Tsuchiya]

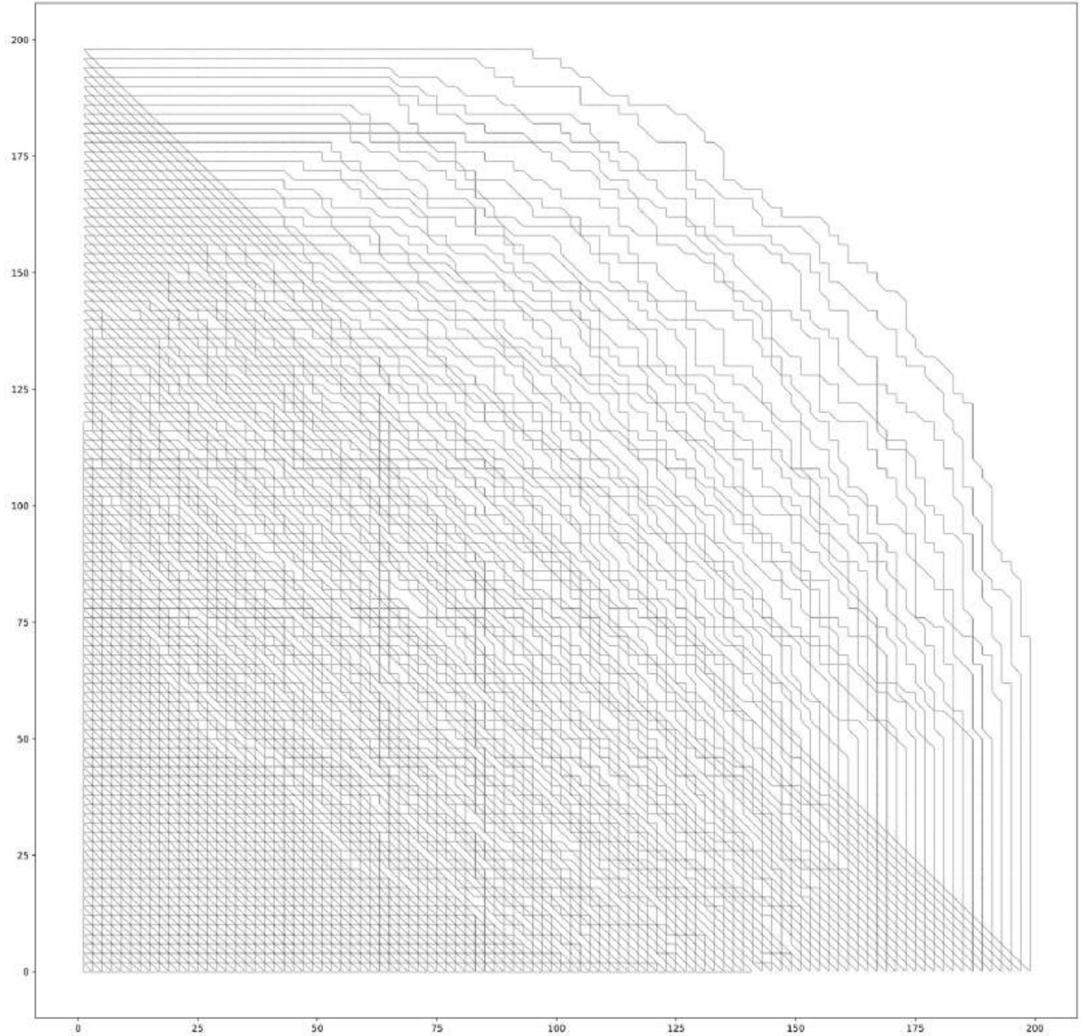
↓

determinant!

6. LIMIT SHAPE : THE ARCTIC PHENOMENON

- large size N ; typical configuration exhibits "frozen" domains / "liquid" domains

$$\begin{array}{ccc} \text{"frozen" domains} & & \text{"liquid" domains} \\ \downarrow & & \downarrow \\ \text{regularly ordered paths} & & \text{disordered paths} \end{array}$$

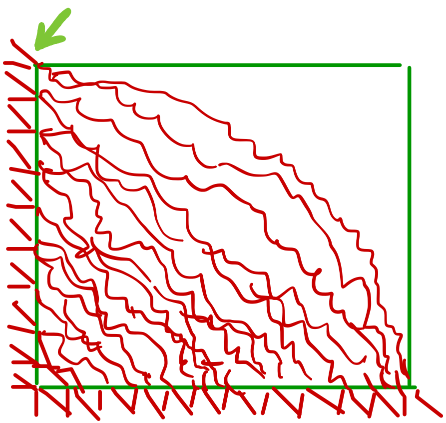


DWBC1
uniform
weights
N = 200

ARCTIC PHENOMENON (20V DWBC1)

- Typical shape of a large configuration

→ use "tangent method" [Colomo - Sportiello '16]

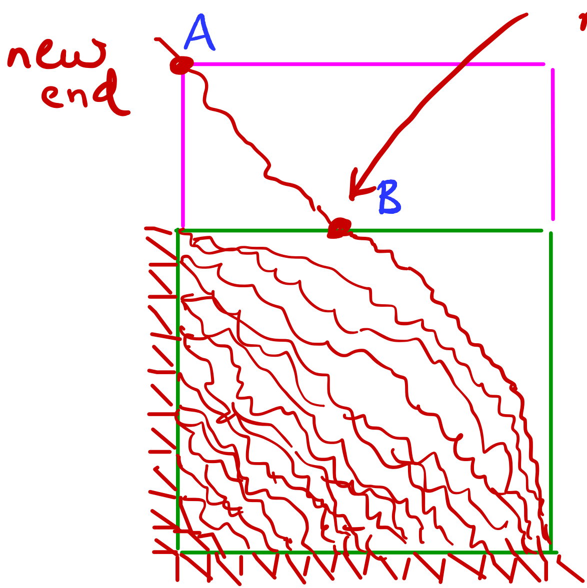


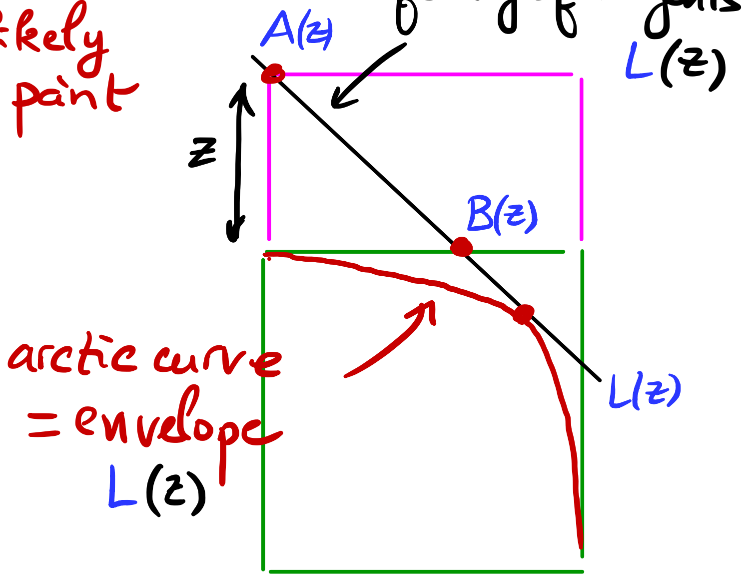
- modify last path exit point
- use this new path as probe for the limit shape

ARCTIC PHENOMENON (20V DWBC1)

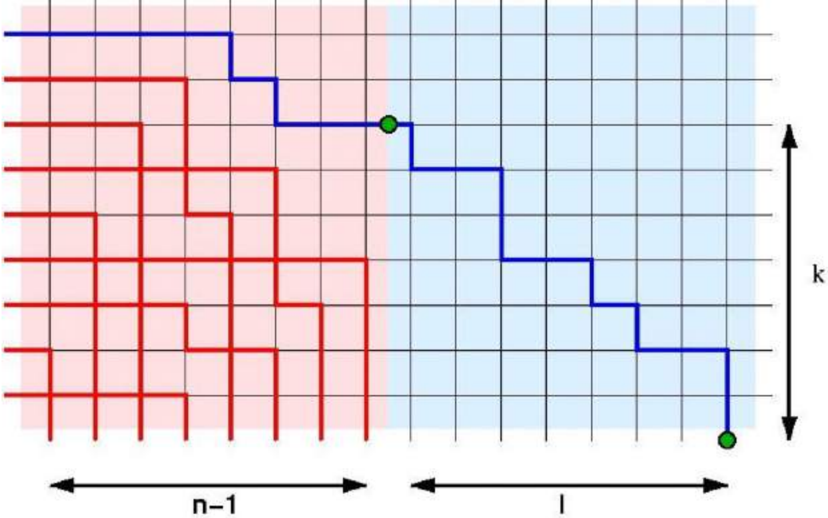
- Typical shape of a large configuration

→ use "tangent method" [Colomo-Sportiello]

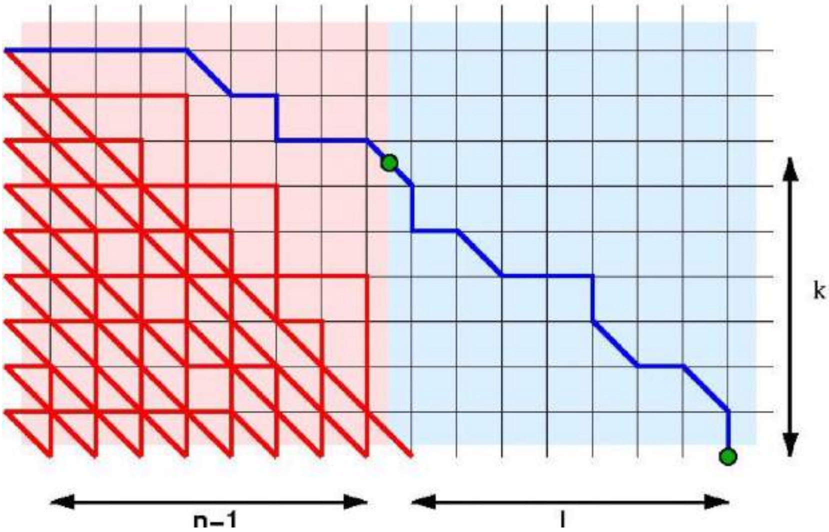




6V DWBC



20V DWBC1



Recipe

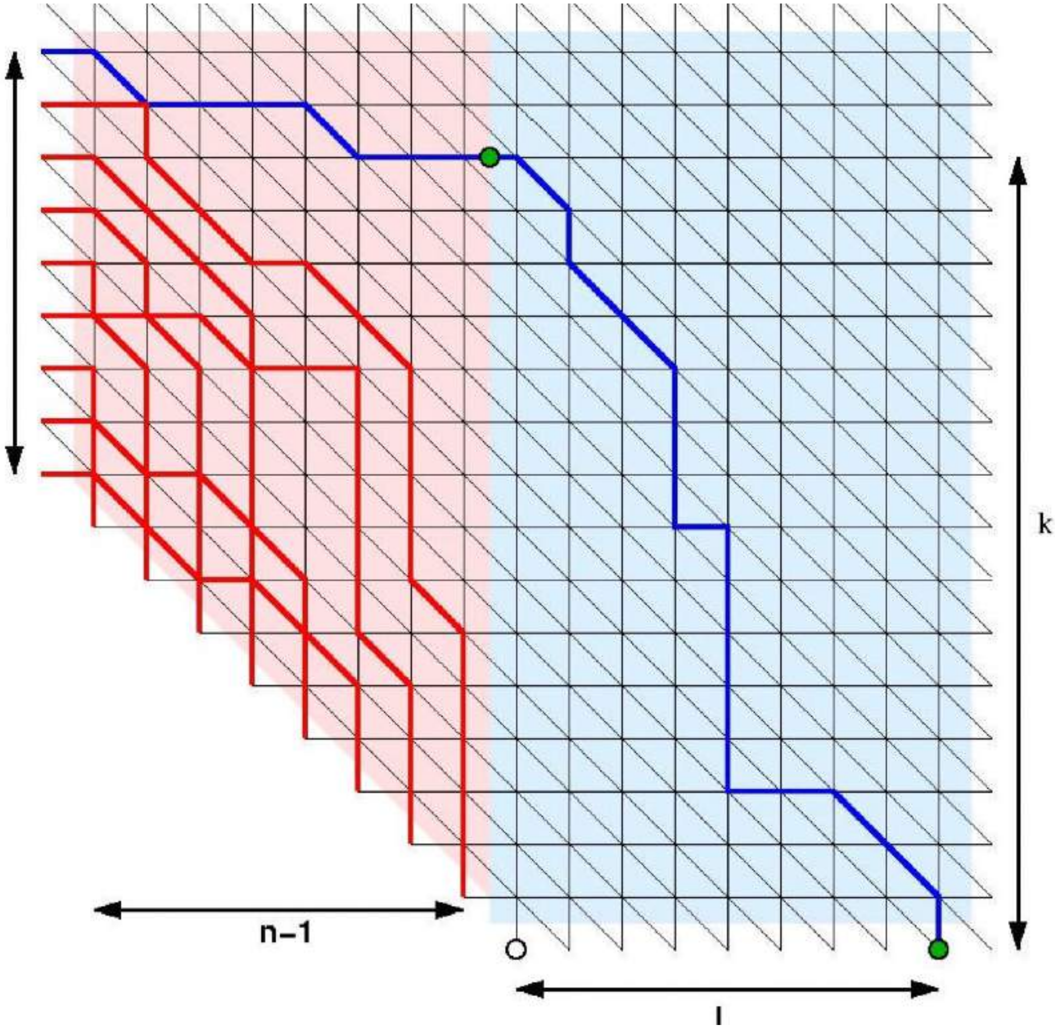
compute both the pink and blue partition functions!

+ large n, ℓ, k estimates

+ saddle-point solution $\Rightarrow k(\ell)$

+ envelope of lines thru $(\ell,0)$ & $(0,k)$

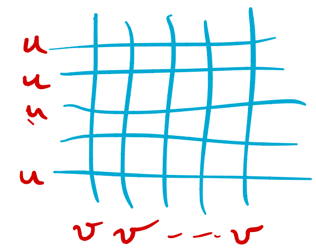
20V DWBC 3
(quadrangle)



- **Partition Function:**

from IK determinant formulas
$(z_i \to u \,,\; w_j \to v)$

$$\frac{Z_{N+1}\, Z_{N-1}}{Z_N^2} + \frac{1}{N^2}\frac{\partial^2}{\partial u\,\partial v}\log Z_N = 0$$

$Z_N =$ 

$$Z_N \simeq e^{-N^2 f} \Rightarrow \boxed{\frac{\partial^2 f}{\partial u\,\partial v} = e^{-2f}}$$

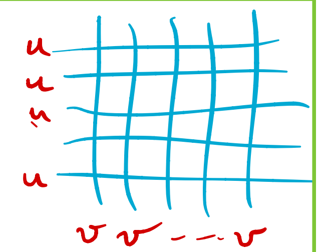
(2D Liouville/Toda eq)

• Partition Function:

from IK determinant formulas

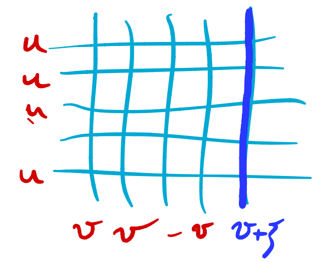
$(z_i \to u\ ,\ w_j \to v)$

$$\frac{Z_{N+1}\,Z_{N-1}}{Z_N^2} + \frac{1}{N^2}\frac{\partial^2}{\partial u\,\partial v}\log Z_N = 0$$

$Z =$ 

$Z_N \simeq e^{-N^2 f} \Rightarrow$ $$\boxed{\frac{\partial^2 f}{\partial u\,\partial v} = e^{-2f}}$$

(2D Liouville/Toda eq)

$Z(\xi) =$ 

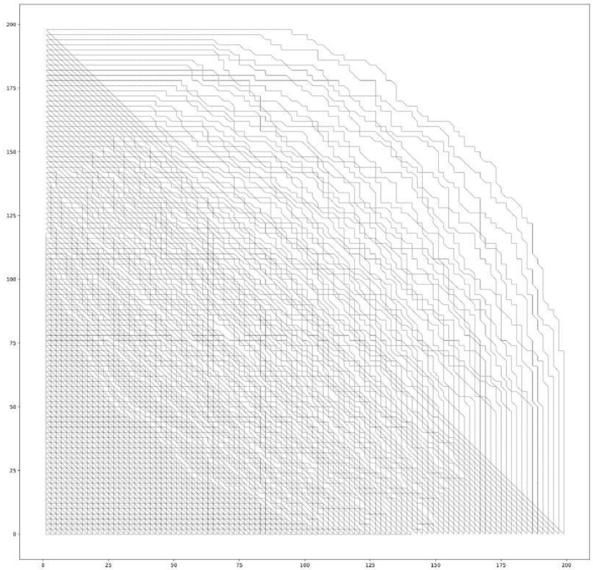
• One-point Function:

$(z_i \to u\ ,\ w_i \to v_{1\le i\le N-1}\ ,\ w_N \to v+\xi)$

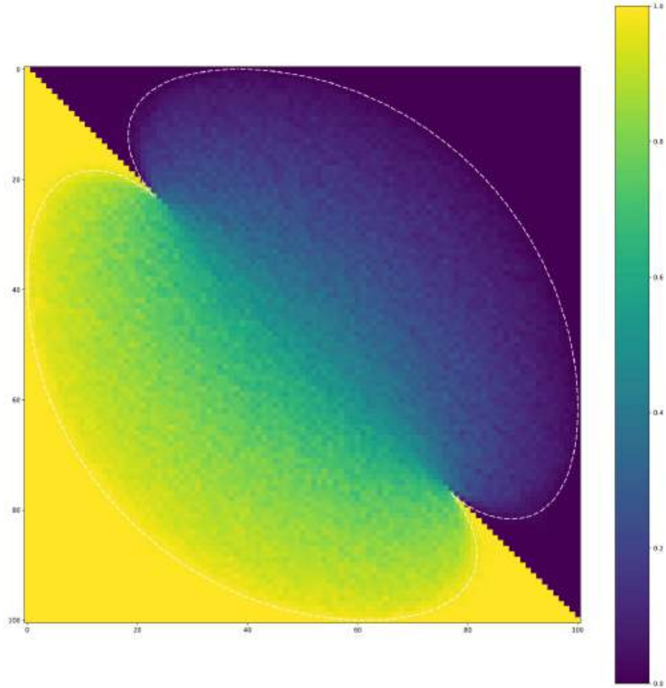
$$H_N = (N-1)!\,\frac{Z_N(\xi)}{Z_N(0)} \Rightarrow \frac{Z_{N+1}\,Z_{N-1}}{Z_N^2}\,\frac{H_{N+1}}{H_N} + \frac{1}{N}\partial_u \log H_N = 0$$

$H_N \simeq e^{-N\psi} \Rightarrow$ $$\boxed{\partial_u \psi = e^{-2f-\psi}}$$

20V-DWBC1 uniform weights



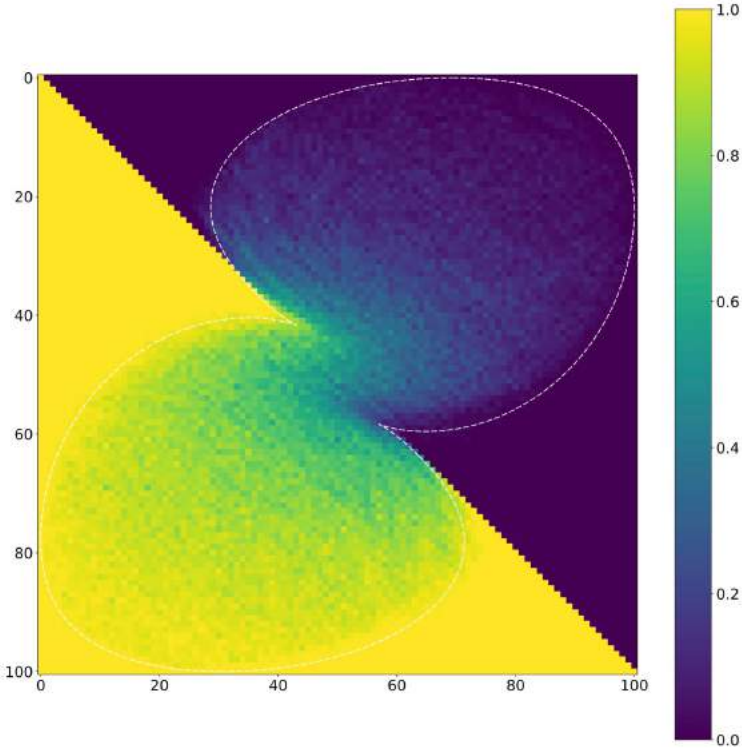
N=200



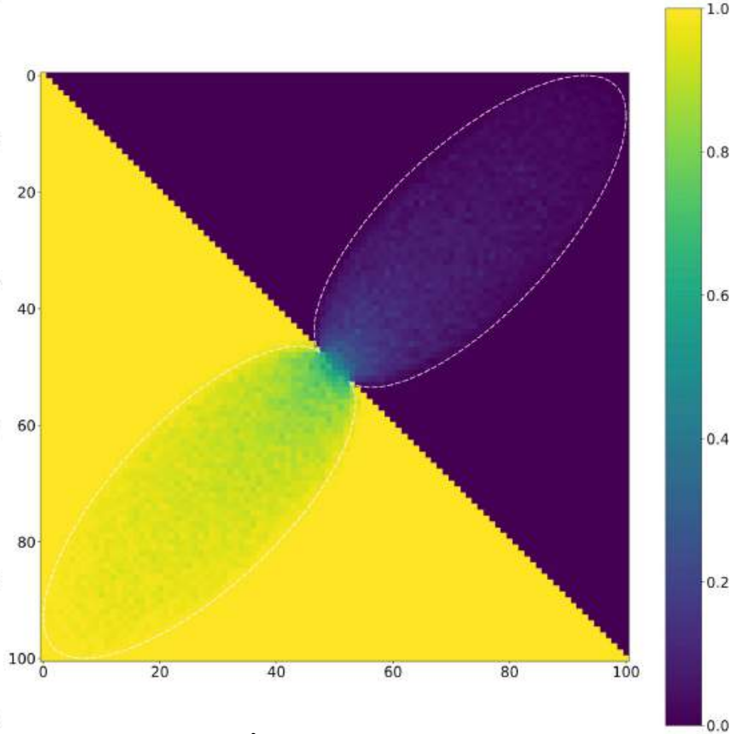
N=100

20V-DWBC1 – Non-uniform integrable weights

$(\omega_0, \omega_1, \ldots, \omega_6)$

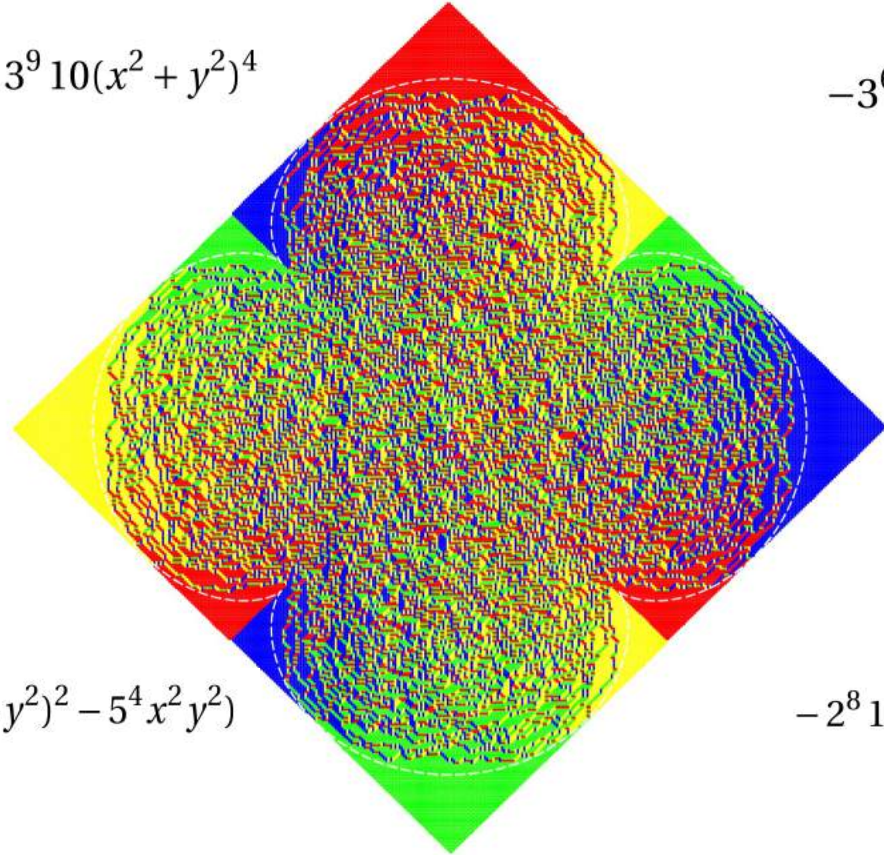


N=100



N=100

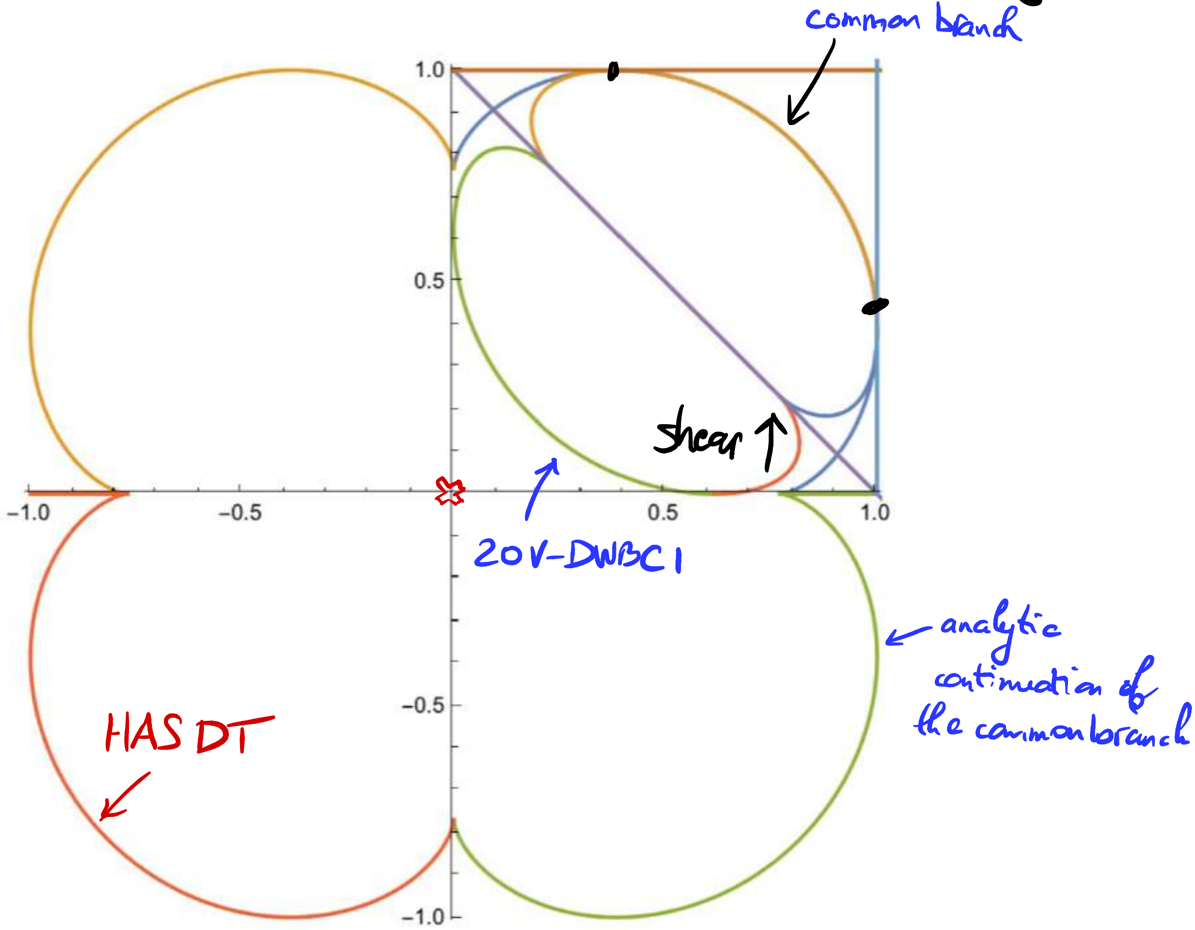
Holey Aztec square domino tilings (uniform weights)



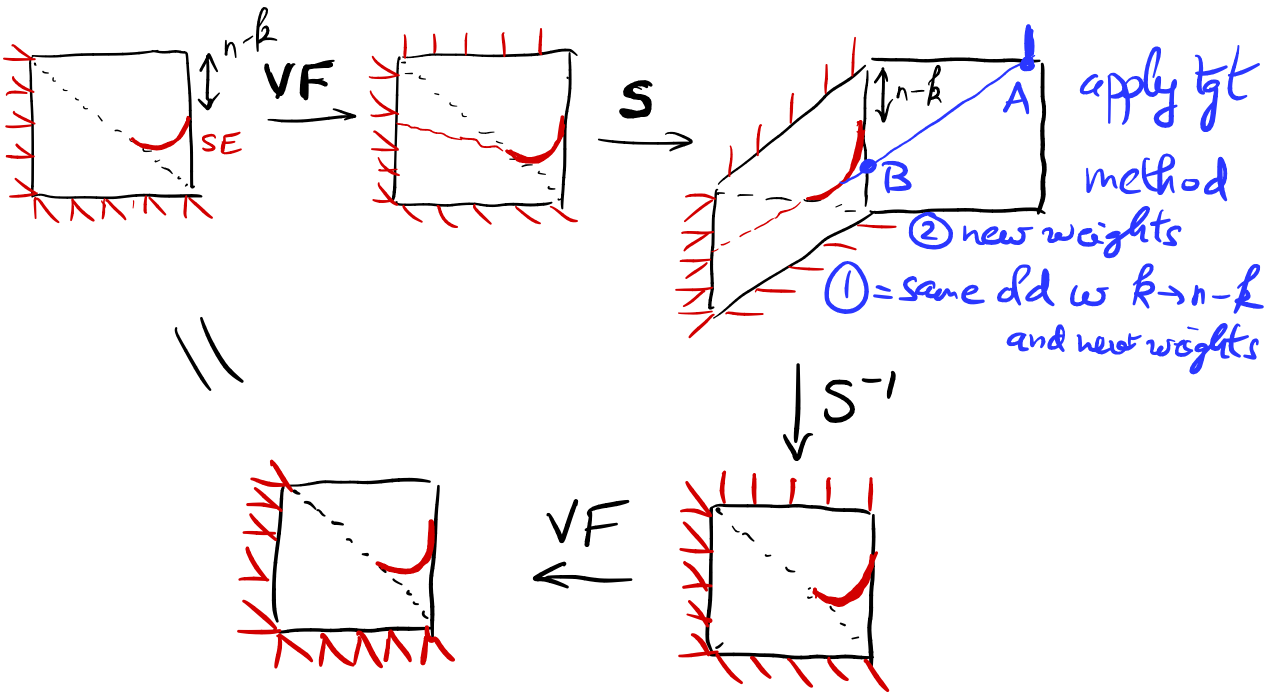
$$3^{11}(x^2+y^2)^5+3^9\,10(x^2+y^2)^4 \quad -3^6\,5(x^2+y^2)^3$$

$$+6^2\,20(73(x^2+y^2)^2-5^4x^2y^2) \quad -2^8\,15(x^2+y^2)-2^{12}=0\,.$$

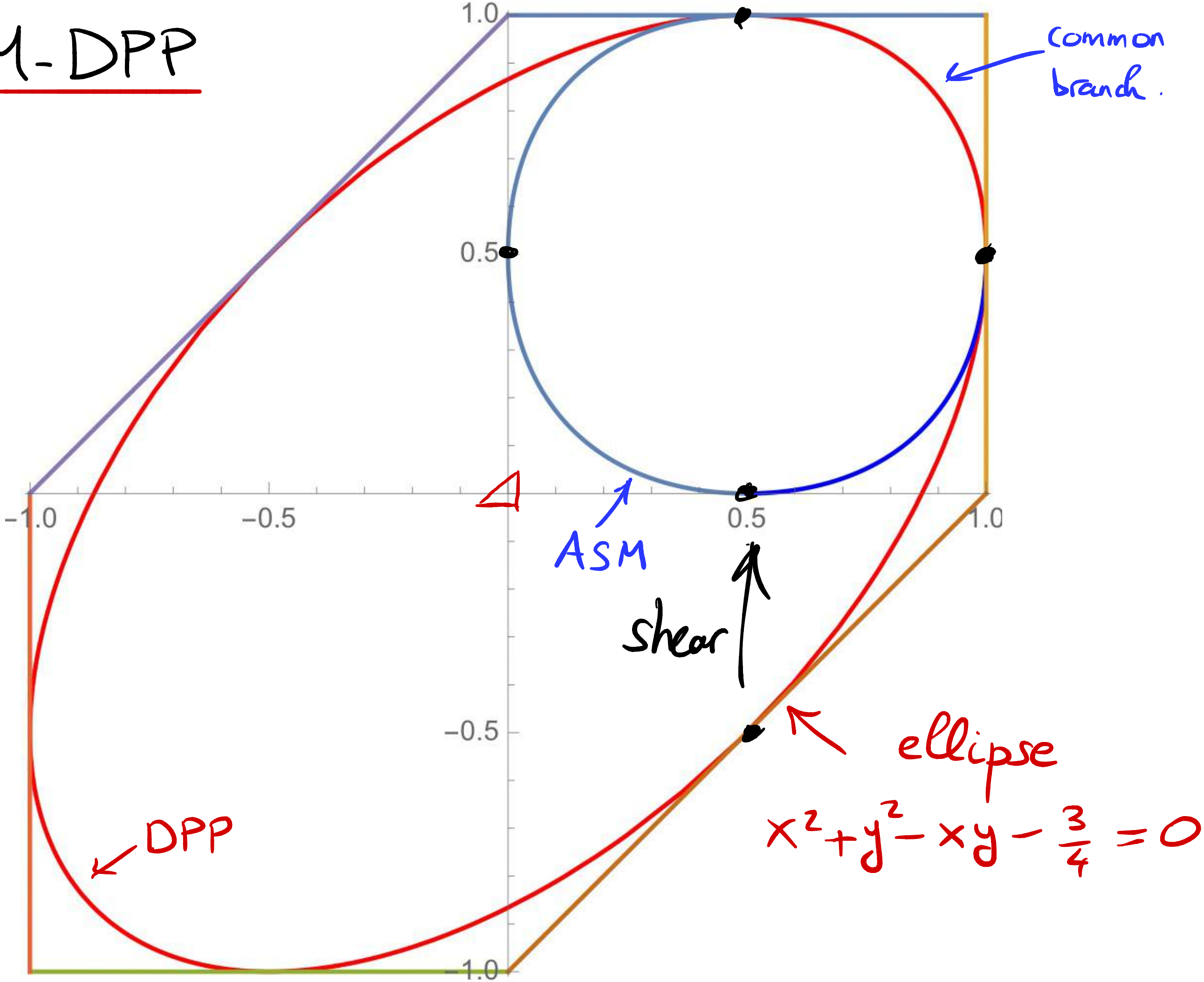
APM – holey Aztec Domino Tiling



THE SHEAR TRICK



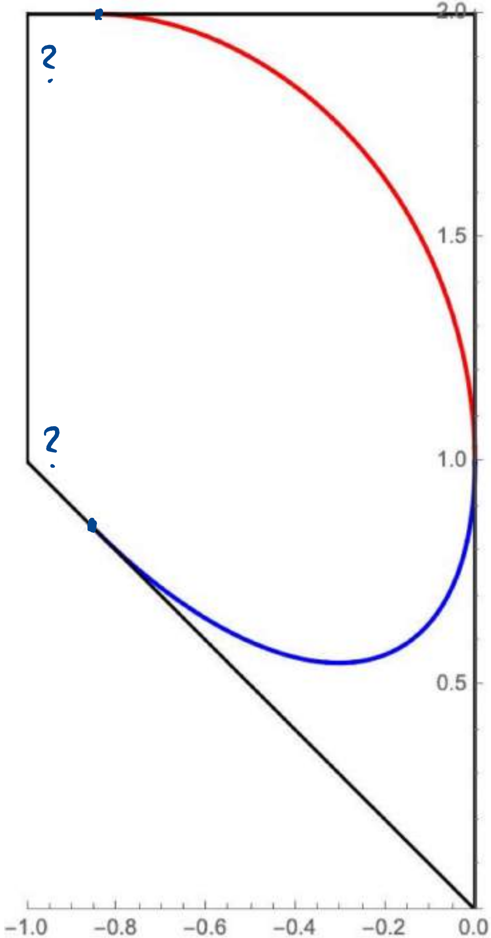
ASM-DPP

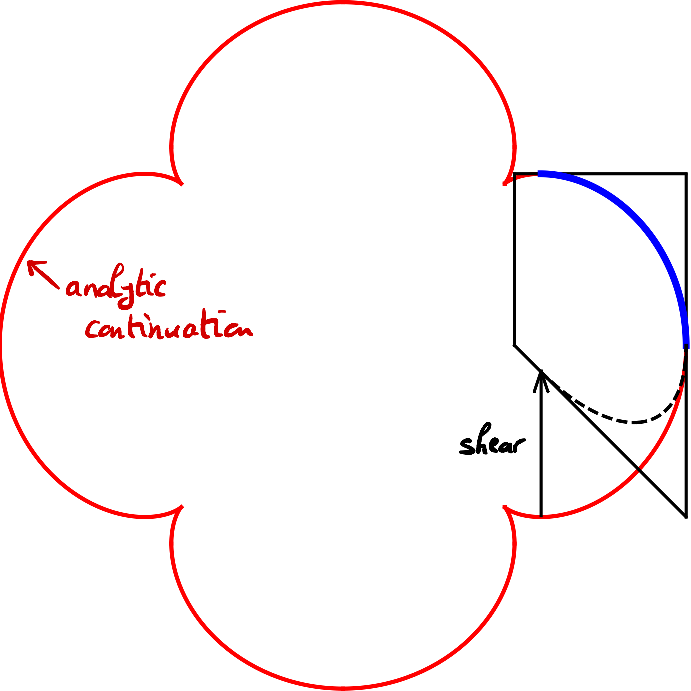


1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0
common branch.
ASM
shear
ellipse
$x^2+y^2-xy-\frac{3}{4}=0$
DPP

20V-DWBC3 on $Q_{n\to\infty}$

open problem

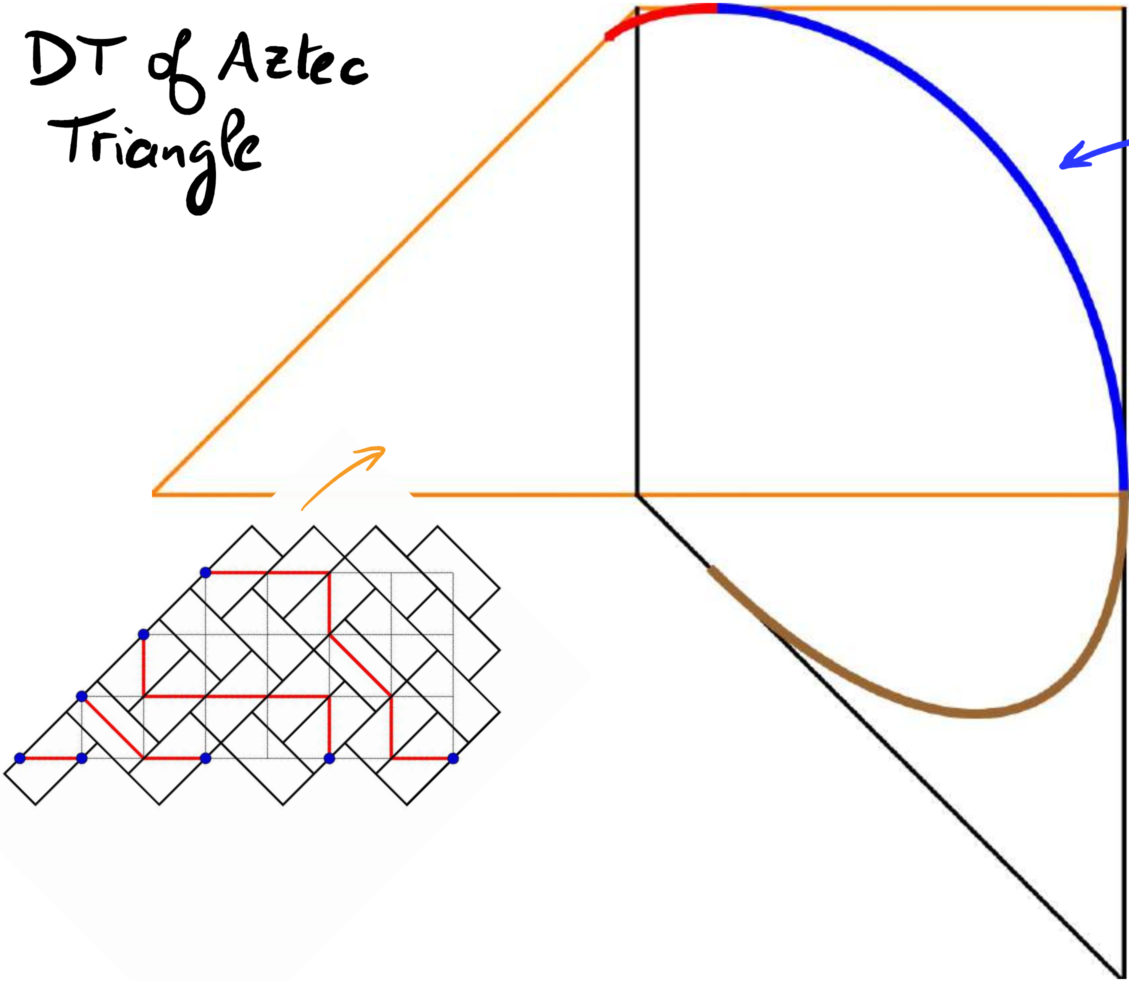




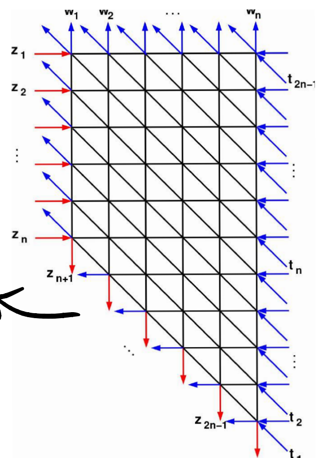
analytic
continuation
shear

DT of Aztec Triangle

common part of Arctic curve



20V DWBC3



7. CONCLUSION

- Triangular ice does have interesting combinatorics!

  osculating lattice paths

- Arctic Phenomenon DWBC 1, 2, 3 have one!

  → use tangent method

  → use refinements and connection to 6V

  U-6V

  Analytic predictions

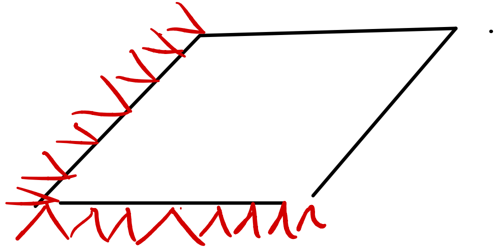
  make them rigorous!

- non-analyticity / shear phenomenon for "interacting fermions".

- related loop models? (webs?) (RS-like conjecture?)
- Classify the "good" boundary conditions?

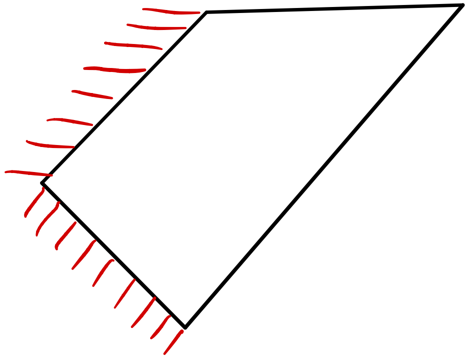
SOLVABLE CASES SO FAR

DWBC 1, 2 (Lozenge)



Z_n^{6V}

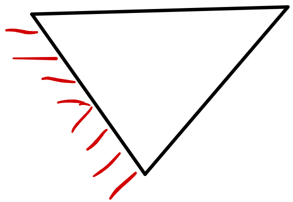
DWBC 3 (quadrangle)



$Z_n^{6V\text{-}U}$

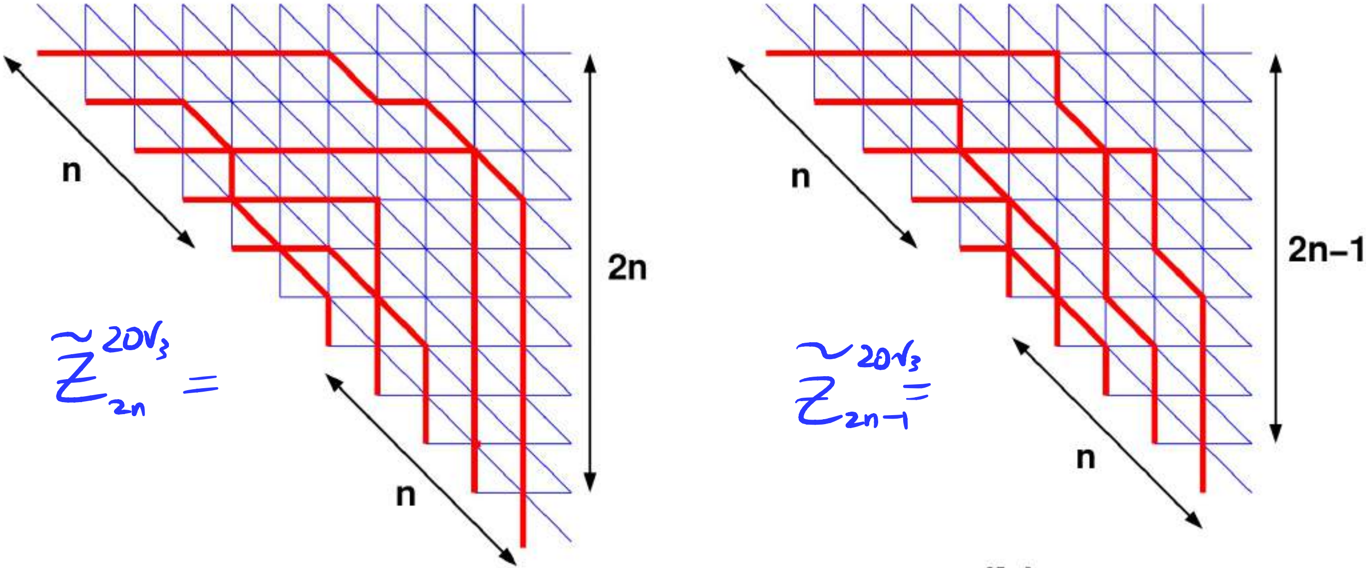
DWBC 3 (triangle)

NEW



$2^{n(n\pm1)/2} \, Z_n^{6V}$

Another 20V DWBC3 model



$\tilde{Z}^{20V_3}_{2n} =$

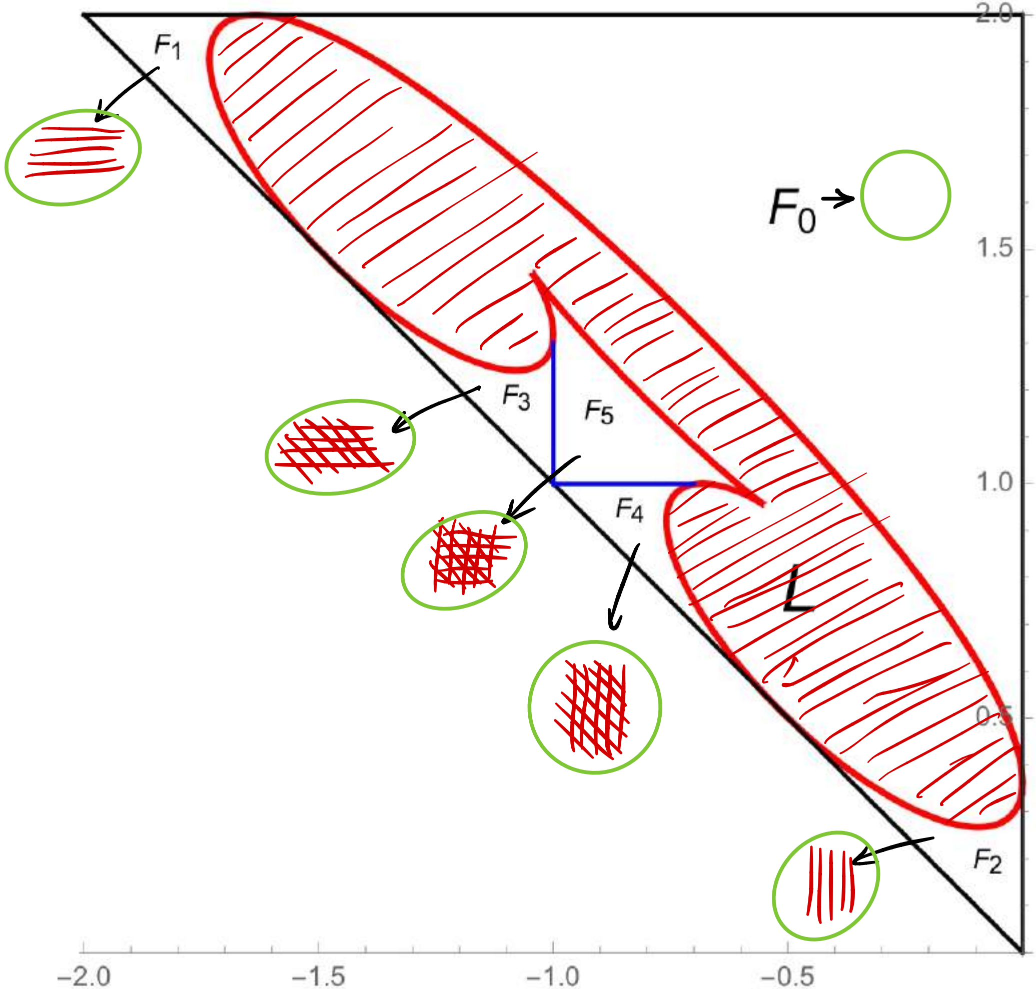
$\tilde{Z}^{20V_3}_{2n-1} =$

(a) (b)

THM

$$\tilde{Z}^{20V_3}_{2n} = 2^{n(n+1)/2} \, Z^{20V\text{-}DWBC_1}_{n}$$

$$\tilde{Z}^{20V_3}_{2n-1} = 2^{n(n-1)/2} \, Z^{20V\text{-}DWBC_1}_{n}$$

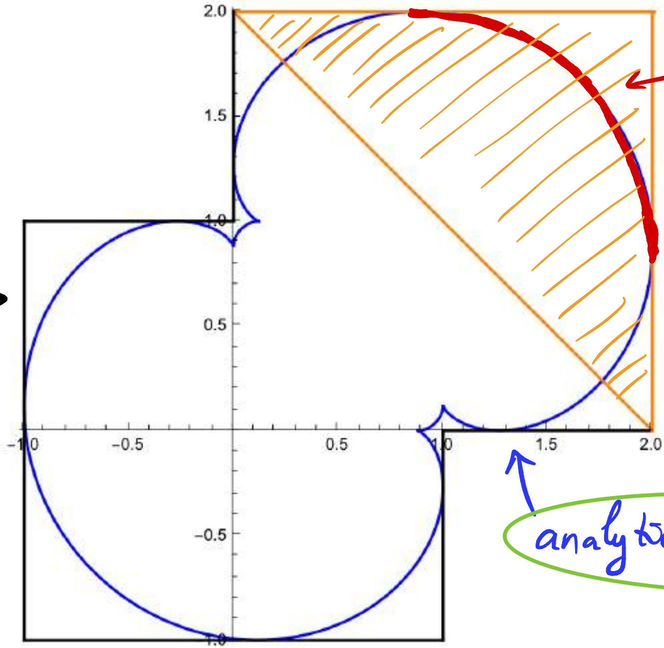


F_1
F_0
F_3
F_5
F_4
L
F_2
2.0
1.5
1.0
0.5
−2.0
−1.5
−1.0
−0.5

OPEN QUESTION: FIND A DOMINO TILING IN BIJECTION WITH $\widetilde{20V}_3$?

Hint: arctic curve for the uniform counting (by tangent method)

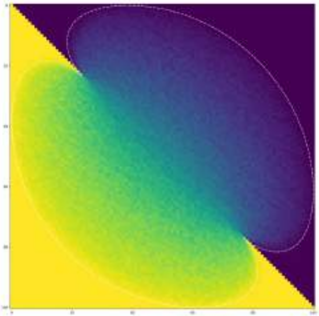
domino tiling ?



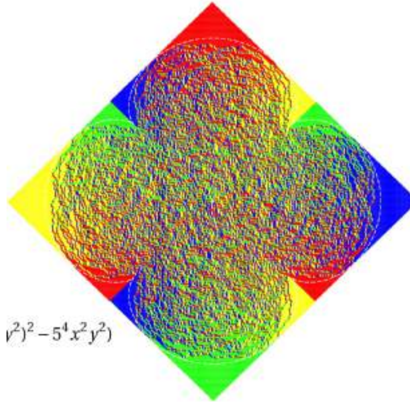
arctic curve

original domain

analytic continuation



Thank You!



<u>Refs:</u>

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- Debin, Di Francesco, Guitter Jour. Stat. Phys. 179 (2020) pp 33–89
- Di Francesco, ArXiv 2102.02920 [math.CO] (2021)
- Di Francesco, ArXiv 2106.02098 [math-ph] (2021)
- P. Di Francesco, in preparation (2022)