TRIANGULAR ICE: COMBINATORICS & LIMIT SHAPES

(PDF + E. Guitter IPHT Saclay)

+ B. Debin UC Louvain

1. ASM, square ice, 6V model and integrability
2. Triangular ice, DWBC, and APM
3. Domino Tilings of the Holey Aztec Square
4. Proof of the APM - HAS DT correspondence
5. Combinatorial Conjectures
6. Limit shape/Arctic Phenomenon
7. Conclusion

A Tale of 3 sequences \[ \{1, 3, 23, 433, 19705, 2151843, \ldots \} \]
\[ \{1, 3, 29, 901, 89893, 28793575, \ldots \} \]
\[ \{1, 4, 60, 3328, 678912, 508035072, \ldots \} \]
ASM and Square Ice

Replace data by dipolar momenta \( \{\rightarrow, \leftarrow, \downarrow, \uparrow\} \)

Ice Rule at each vertex

\# incoming arrows = \# outgoing arrows \( \Rightarrow 6V \)
ASM and 6V model

Replace data by dipolar momenta \{\rightarrow, \leftarrow, \downarrow, \uparrow\}

Ice Rule at each vertex

\# incoming arrows = \# outgoing arrows \Rightarrow 6V

+ Domain Wall Boundary Conditions

(n \times n square)
Bijections

1. 6V configs
2. Osculating paths
   \((NW \rightarrow SE)\)
3. ASM entries

Transmitter vertices
\[\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[0 \quad 0 \quad 0 \quad 0 \quad 0 \]

Reflector vertices
\[\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[+1 \quad -1 \]

Alternance conditions

odd # of reflections!
$$\text{ASM}_n = \frac{\prod_{i=0}^{n-1} (3i+1)!}{\prod_{i=0}^{n+1} (n+i)!}$$

INTEGRABILITY

- Boltzmann weights

$R$ operator

$\dim V_i = 2$
$V_1 = \langle \rightarrow, \leftarrow \rangle \exists \alpha$
$V_2 = \langle \uparrow, \downarrow \rangle \exists \beta$

Matrix entries in $\alpha \otimes \beta \rightarrow \beta \otimes \alpha$

6 non-zero entries out of 16 (i.e., rule).

\[ a \quad b \quad c \]
One can pick "integrable weights" such that:

\[ a(z, w) = z - w \]
\[ b(z, w) = q^{-2} z - q^{2} w \]
\[ c(z, w) = (q^{2} - q^{-2}) \sqrt{zw} \]

\( \dim V = 2 \) \ TRIGONOMETRIC R-matrix of 6V MODEL
IZERGIN–KOREPIN Determinant

\[
\mathcal{Z}_{6N} \left[ \begin{array}{cccc}
W_1 & \cdots & W_N \\
Z_1 & \cdots & Z_N
\end{array} \right] = \prod_{i,j=1}^{N} a(z_i, w_j) b(z_i, w_j) \prod_{1 \leq i < j \leq N} (Z_i - Z_j)(W_i - W_j) \\
\times \det \left( \frac{c(z_i, w_j)}{a(z_i, w_j) b(z_i, w_j)} \right)_{1 \leq i, j \leq N} \\
\sqrt{V(z_i, w_j)} \ [\text{cf. Lamers' talk}]
\]
2. TRIANGULAR ICE (20V model)

ice rule

\[ \text{at each vertex} \quad \left( \frac{6}{3} \right) = 20 \]

[Kelland, Baxter]
TRIANGULAR ICE (20V model)

Twenty vertices:

Osculating Schröder paths

h, v, d steps
DOMAIN WALL BOUNDARY CONDITIONS

DWBC1

DWBC2
Numbers of Configurations on an $n \times n$ grid:

**DWBC 1,2**

$$A_n = 1, 3, 23, 433, 19705, 2151843, ...$$

**DWBC 3**

$$B_n = 1, 3, 29, 901, 89893, 28793575, ...$$

[computed by transfer matrix]
$20^v$

DWBC1

configurations

$n = 3$

$(23)$
3. **Domino Tilings of the Holey Square**

**With Quarter-Turn Symmetry**

Domino Tilings: use $\diamondsuit$ and $\boxdot$ $2 \times 1$ dominoes

Rotational symmetry by $\frac{\pi}{2}$

NB: the hole makes it tileable!
DOMINO TILINGS OF THE HOLEY SQUARE
WITH QUARTER-TURN SYMMETRY

\[ S_n \]

sample tiling
Counting Configurations

- Non-intersecting Schröder paths with fixed ends
- First step cannot be horizontal
- Start and ends identified (cone)
Counting Configurations

Thm [PDF-Gutier 19]

\[ T_4(S_n) = \det \left( \sum_{0 \leq i,j \leq n-1} \frac{1}{1-z^i} + \frac{2z}{(1-z)(1-z-w-zw)} z^{i,j} \right) \]

\textbf{Proof (Cayley-Binet)} \quad \det (\text{Id} + M) = \sum_{i_1 < \cdots < i_k} |M|_{i_1 \cdots i_k} \quad \text{(Gessel-Viennot)}

\[ T_4(S_n) = 1, 3, \underline{23}, 433, 19705, 2151843, \ldots \]

\textbf{Ex:} \quad n=3 \quad \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 4 & 8 & 12 \end{bmatrix} = \boxed{23} \quad \text{Domino Tiling Configurations}
$n = 3$ (holey square) Domino Tilings
(n=3  DWBC1  20V configurations)
4. **PROOF OF THE CORRESPONDENCE WITH 20V - DWBC 1,2**

idea

- use integrable weights for the 20V
- deform the line arrangement into a 6V
- use 6V results (Izergin-Korepin def.)
- refinement
ICE MODEL ON THE KAGOME LATTICE

\[ t_1, t_2, t_3, w_1, w_2, w_3 \]

\[ 20V \text{ weights} \quad 6V \text{ weights on } 3 \text{ sublattices } 133 \]

Triangular lattice ice

Kagome Lattice ice
**BOLTZMANN WEIGHTS**

1. $w^1 = a_1 b_1 c_1$
2. $w^2 = a_2 b_2 c_2$
3. $w^3 = a_3 b_3 c_3$

Weights of the 6V models on the 3 sublattices.

- 6V weights are given by sums over inner triangle configs.

Example: $w = a_1 b_2 c_3 + c_1 c_2 b_3 = b_1 a_2 c_3$

- Homogeneous case: 3 parameter family: $(z, t, q)$
Homogeneous weights:

\[ q = e^{i\eta} \]
\[ z = e^{i(\eta + \lambda)} \]
\[ w = e^{-i(\eta + \lambda)} \]
\[ t = e^{i\mu} \]

\[
\omega_0 = \sin(\lambda + \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)
\]
\[
\omega_1 = \sin(\lambda - \eta) \sin\left(\frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)
\]
\[
\omega_2 = \sin(2\eta) \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)
\]
\[
\omega_3 = \sin(2\eta)^3 + \sin(\lambda + \eta) \sin\left(\frac{\lambda - \eta + \mu}{2}\right) \sin\left(\frac{\lambda - \eta - \mu}{2}\right)
\]
\[
\omega_4 = \sin(2\eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda + 3\eta - \mu}{2}\right)
\]
\[
\omega_5 = \sin(2\eta) \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right)
\]
\[
\omega_6 = \sin(\lambda - \eta) \sin\left(\frac{\lambda + 3\eta + \mu}{2}\right) \sin\left(\frac{\lambda - \eta - \mu}{2}\right),
\]
Remark: uniform weights \( \omega_i = 1 \ \forall i \) are obtained for:
\[
\eta = \frac{\pi}{8} \quad \lambda = \frac{5\pi}{8} \quad \rho = 0
\]

This corresponds to non-uniform weights on the 3 sublattice 6V models (up to an overall factor).

| ① | \( a_1 = 1 \quad b_1 = \sqrt{2} \quad c_1 = 1 \) |
|---|---|---|
| ② | \( a_2 = \sqrt{2} \quad b_2 = 1 \quad c_2 = 1 \) |
| ③ | \( a_3 = \sqrt{2} \quad b_3 = 1 \quad c_3 = 1 \) |
TRANSFORMATION INTO a 6V MODEL

20V DWBC-2 (integrable weights).
6V DWBC (sublattice 1 only).
Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights $= 1$ is equal to that of the 6V model with weights $(a, b, c) = (1, \sqrt{2}, 1)$ and DWBC.
A lattice of KAGOME (Daikokuya, Kitashirakawa)
Example of size $n=3$

$20V$-DWBC1 vs $6V$ aka ASM

$\times \sqrt{2}$

$\times \sqrt{2}$

(bweights)

Total = 23 APM of size 3

$\left\{ \begin{array}{c}
(100) \\
(010)
\end{array} \right\} \left\{ \begin{array}{c}
(100) \\
(010)
\end{array} \right\} \left\{ \begin{array}{c}
(010) \\
(100)
\end{array} \right\} \left\{ \begin{array}{c}
(010) \\
(1-11)
\end{array} \right\}$

7 ASM of size 3.
Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights $= 1$ is equal to that of the 6V model with weights $(\alpha, \beta, \gamma) = (1, \sqrt{2}, 1)$ and DWBC.

Then use classical result by Korepin-Izergin for the 6V-DWBC and spectral parameters $(z_1, \ldots, z_n, w, \ldots, w_n)$.

$$Z_{6V-DWBC}(z_1, \ldots, z_n, w, \ldots, w_n) = \frac{\prod_{i=1}^n C(z_i, w_i) \prod_{i,j=1}^n a(z_i, w_j) b(z_i, w_j)}{\prod_{1 \leq i < j \leq n} (z_i - z_j)(w_i - w_j) \det_{i,j=1}^n \frac{1}{a(z_i, w_j) b(z_i, w_j) \text{det}} \{ a(z_i, w_j) b(z_i, w_j) \}}$$

→ Limiting procedure → same det as holey square DT!

→ Refinements (of Behrend, PDF, Zinn Justin) .
5. The DWBC 3 Conjectures

\[ b_n = 1, 3, 29, 901, ... \]
The DWBC Conjectures

[OEIS for $B_n$] $\rightarrow$ Domino Tilings of a $2n \times 2n$ square

$= 2^n b_n^2$

$b_n = 1, 3, 29, 901, \ldots$

[Paalman] proof of integrality of $b_n$: found a Domino Tiling interpretation

[see also Ciucu]

$2n$

$\begin{align*}
\sum_{i=1}^{2n-1} & \\
& = \prod_{1 \leq i < j \leq 2n} \left( 4 \cos \frac{2\pi i}{2n+1} + 4 \cos \frac{2\pi j}{2n+1} \right)
\end{align*}$
Conjecture 1. [PDF-E Quitter 19] The configurations of the 20V-DWBC3 model on an n x n grid are counted by the Domino Tilings of Baxter’s triangle.

But we can do better....
Pentagon of triangular ice
"raise the roof" above Patchler's triangle
Conjecture 2. [PDF+ E. Guitter 19] The number of configurations of triangular ice in a pentagon \( 25/\text{DWBC3} \) is equal to that of domino tilings of Patashnik's raised triangle.

\[
\text{Conj.} \quad 2n-1
\]

\[
\text{Ice} \quad \text{interacting fermions}
\]

\[
\text{Domino} \quad \text{free fermions}
\]
20V-DWBC 3 on $Q_n$

$Z_n = 1, 4, 60, 3328, \ldots$
**Theorem** [DiFrancesco 2021]

\[
Z^{Z_{\text{Q}_n}}_{\text{Q}_n} = Z^{\text{DT}}_{\text{AT}_n} = \det \left( \frac{1+u}{(1-uv)^2-u(1+u)^2} \right)_{\alpha_\delta, \beta_\delta \leq n-1}
\]

**Conjecture 3**

\[
Z^{Z_{\text{Q}_n}}_{\text{Q}_n} = 2^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}
\]

\[= 1, 4, 60, 3328, 678912, \ldots\]

+ refinements

Progress

[CIUCU '21] [Krattenthaler '21]
[Trubitsin + PDD '21]
• Proof of Thm Along the same lines

20V-DWBC3 on $Q_n$
6. \textbf{LIMIT SHAPE: THE ARCTIC PHENOMENON}

- \textit{large size N}; \textit{typical configuration exhibits "frozen" domains / liquid" domains}

\begin{itemize}
  \item \textit{regularly ordered paths}
  \item \textit{disordered paths}
\end{itemize}
DWBC1 uniform weights $N = 200$
ARCTIC PHENOMENON (20V DWBC1)

• Typical shape of a large configuration
  → use “tangent method” [Colomo-Sportiello ’16]

• modify last path exit point
  • use this new path as probe for the limit shape
ARCTIC PHENOMENON (20V DWBC1)

- Typical shape of a large configuration
  → use “tangent method” [Colomo-Sportelli]

new end
A
B

most likely exit paint

arctic curve = envelope $L(z)$

family of tangents

L(z)

L(z)
Recipe compute both the pink and blue partition functions!

- large $n, l, k$ estimates
- saddle-point solution
  \[ k(e) \]
- envelope of lines thru $(\xi_0, k)$
ZOVDWBC 3
(quadrangle)
Partition Function:

From 1K determinant formulas

\[ Z_{n+1} \frac{Z_{n-1}}{Z_n^2} + \frac{1}{N^2} \frac{\partial^2}{\partial u \partial v} \log Z_n = 0 \]

\[ Z_n = e^{-N^2 \phi} \Rightarrow \]

\[ \frac{\partial^2 \phi}{\partial u \partial v} = e^{-2\phi} \]

(2D Liouville/Toda eq)
- **Partition Function:** from 1K determinant formulas
  \[
  \frac{Z_{N+1} Z_{N-1}}{Z_N^2} + \frac{1}{N^2} \frac{\partial^2}{\partial u \partial \nu} \log Z_N = 0
  \]
  \[Z_N = e^{-N^2 \psi} \Rightarrow \frac{\partial^2 f}{\partial u \partial \nu} = e^{-2 \psi} \]
  (2D Liouville/Toda eq)

- **One-particle Function:**
  \[
  H_N = (N-1)! \frac{Z_N(1)}{Z_N(0)} \Rightarrow \frac{Z_{N+1} Z_{N-1}}{Z_N^2} \frac{H_{N+1}}{H_N} + \frac{1}{N} \partial_u \log H_N = 0
  \]
  \[H_N = e^{-N^2 \psi} \Rightarrow \partial_u \psi = e^{-2 \psi - \psi} \]
20V-DWBC1, uniform weights

$N = 200$

$N = 100$
20 V-DWBC 1 - Non-uniform integrable weights 

\((\omega_0, \omega, \ldots, \omega_6)\)
Holy Aztec square domino tilings (uniform weights)

\[3^{11}(x^2 + y^2)^5 + 3^9 10(x^2 + y^2)^4 - 3^6 5(x^2 + y^2)^3\]

\[+ 6^2 20(73(x^2 + y^2)^2 - 5^4 x^2 y^2) - 2^8 15(x^2 + y^2) - 2^{12} = 0\]
APM - holey Aztec Domino Tiling

HAS DT

20V-DWBC1

shear ↑

analytic continuation of the common branch

common branch

APM-holeyn-ztecdomino.li/ing-.
THE SHEAR TRICK

\[ \uparrow \downarrow \rightarrow VF \rightarrow S \rightarrow S^{-1} \]

1. Same old weights \( k \rightarrow n-k \)
2. New weights

apply tgt method
ASM-DPP

\[ x^2 + y^2 - xy - \frac{3}{4} = 0 \]
on $Q_{n \to \infty}$

(open problem)
analytic continuation

shear
DT of Aztec Triangle

.common part of Arctic curve

20V DWBC3
7. CONCLUSION

- Triangular ice does have interesting combinatorics!
  - Osculating lattice paths

- Arctic Phenomenon DWBC 1, 2, 3 have one!
  - Use tangent method
  - Use refinements and connections to 6V
  - Analytic predictions make them rigorous!

- Non-analyticity/shear phenomenon for "interacting fermions".

- Related loop models? (webs?) (RS-like conjecture?)

- Classify the "good" boundary conditions?
SOLVABLE CASES SO FAR

**DWBC1, 2**  
(logenge)

**DWBC3**  
(quadrangle)

**NEW**

**DWBC3**  
(triangle)
Another 20V DWBC3 model

\[
\tilde{Z}_{2n}^{20V_3} = 2^{n(n+1)/2} \tilde{Z}_n^{20V-DWBC_1}
\]

\[
\tilde{Z}_{2n-1}^{20V_3} = 2^{n(n-1)/2} \tilde{Z}_n^{20V-DWBC_1}
\]
OPEN QUESTION: FIND A DOMINO TILING IN BIJECTION WITH $20V_3$?

Hint: arctic curve for the uniform counting (by tangent method)
Thank You!

Refs:

• Di Francesco, ArXiv 2102.02920 [math.CO] (2021)
• Di Francesco, ArXiv 2106.02058 [math-ph] (2021)
• P. Di Francesco, in preparation (2022)