Open Problems in Algebraic Combinatorics

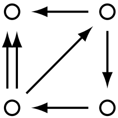
Quiver mutations

Sergey Fomin (University of Michigan)

Quivers

Definition

A quiver is a finite oriented graph. Multiple edges are allowed. We do not allow oriented cycles of length 1 or 2.



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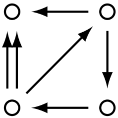

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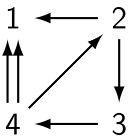
Exchange matrices

To a quiver Q with vertices labeled $1, 2, \ldots, n$, we can associate an $n \times n$ skew-symmetric exchange matrix $B = B(Q)$:

Q



$$B(Q) = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \end{bmatrix}$$

Conversely, the matrix $B(Q)$ determines the quiver Q.

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Most topics discussed today can be extended from quivers to skew-symmetrizable matrices.

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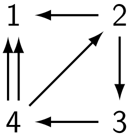

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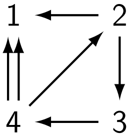

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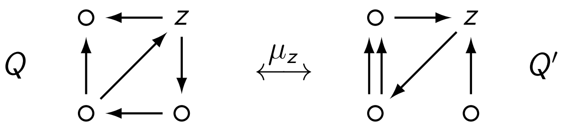

Definition

The *quiver mutation* $\mu_z : Q \mapsto Q'$ is computed in three steps.

1. For each instance of $x \to z \to y$, introduce an edge $x \to y$.
2. Reverse the direction of all edges incident to z.
3. Remove oriented 2-cycles, one by one.

Observation

Quiver mutations are involutions: $\mu_z(\mu_z(Q)) = Q$.

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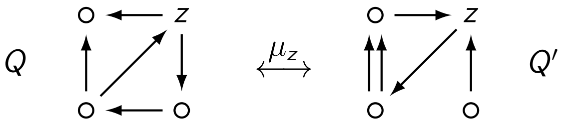

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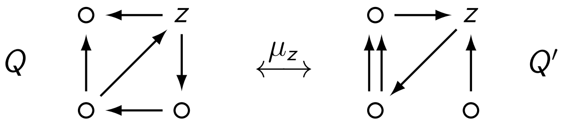

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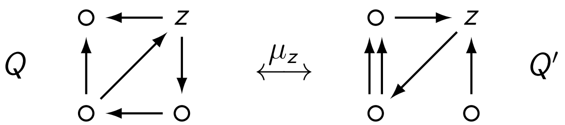

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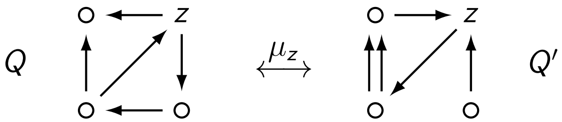

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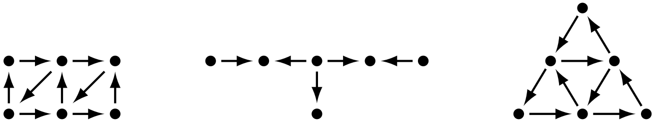
Mutation equivalence



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Two quivers Q and Q' are called mutation equivalent if they are related to each other by a sequence of mutations and isomorphisms.
Notation: $Q \sim Q'$.

Definition

We denote by $[Q]$ the mutation equivalence class of a quiver Q. Thus $[Q] = [Q']$ if and only if $Q \sim Q'$. We say that Q has finite mutation type if the mutation class $[Q]$ is finite.

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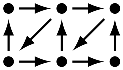 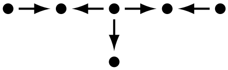 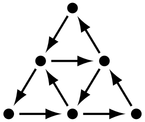

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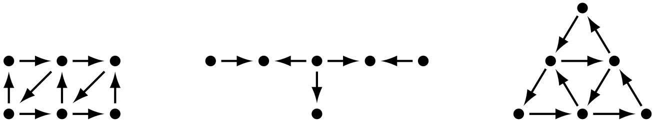

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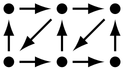 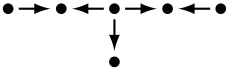 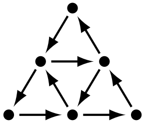

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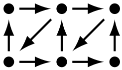
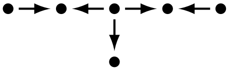
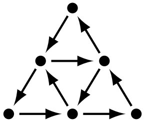

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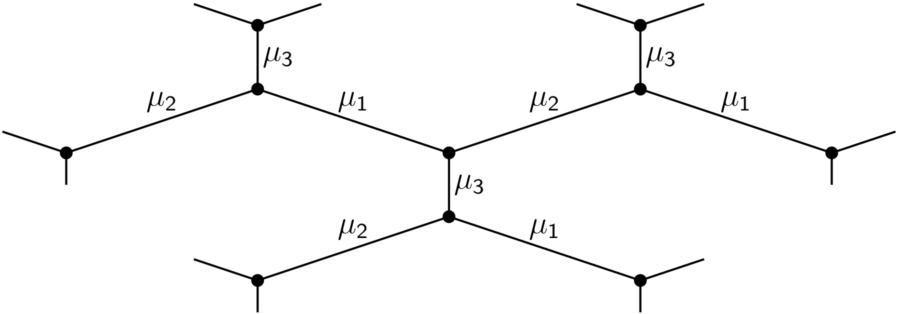
Cluster algebras

Each mutation class defines a cluster algebra.



Detecting mutation equivalence

Problem

Determine whether two given quivers are mutation-equivalent.

- No algorithm for solving this problem is known for any $n \geq 4$.
- For $n = 3$, the problem can be solved using a descent algorithm [Assem, Blais, Brüstle, Samson 2006] that identifies a canonical "minimal representative" in a given mutation class.

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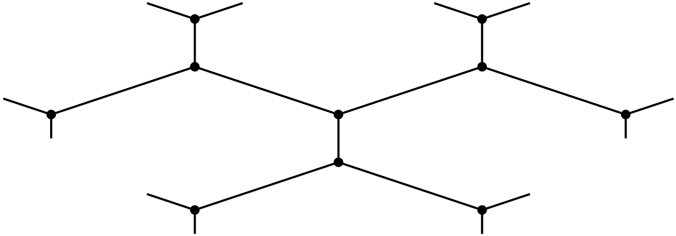

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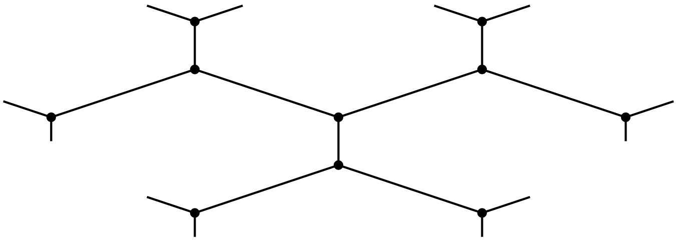

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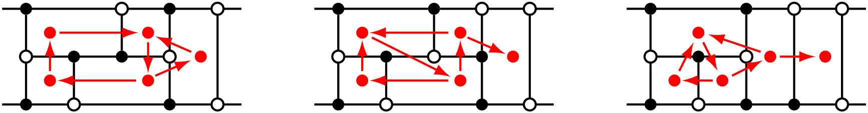
Plabic graphs [Postnikov 2006]

Any planar bicolored (plabic) graph P gives rise to a quiver $Q(P)$.



Postnikov's local moves translate into quiver mutations.

M. Shapiro's conjecture

For quivers associated with plabic graphs, mutation equivalence can be alternatively described by taking the transitive closure of the relation coming from Postnikov local moves on plabic graphs.

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Determine whether two given plabic graphs are move-equivalent.

Solved for reduced plabic graphs [Postnikov].

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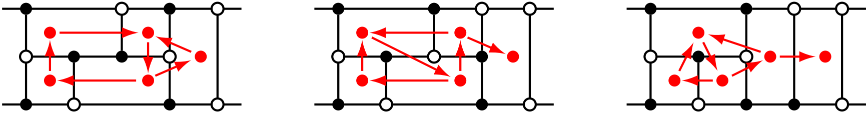

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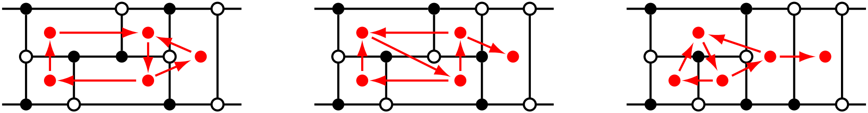

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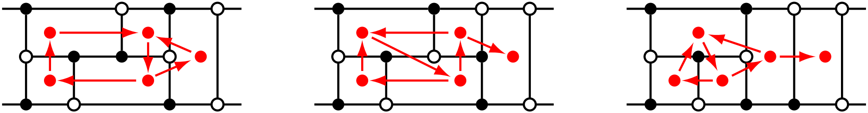

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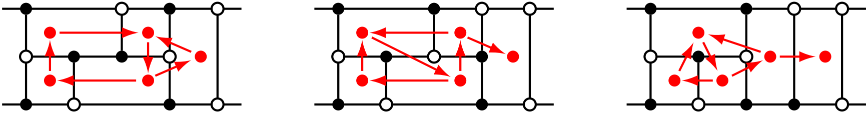

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Invariants of quiver mutations

Such invariants can be used to show that some quivers are not mutation equivalent.

Known invariants

- number of vertices in a quiver;
- rank and determinant of $B(Q)$ [Berenstein, SF, Zelevinsky 2005];
- gcd of the matrix entries in each column of $B(Q)$;
- topological invariants of cluster varieties [Lam, Speyer 2017+];
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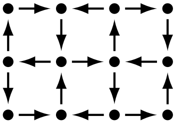
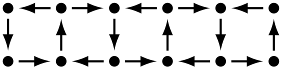


3×4 grid quiver



2×6 grid quiver

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Is there an invariant of quiver mutations that distinguishes between such quivers?

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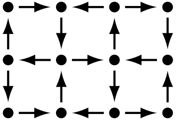

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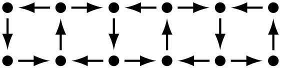

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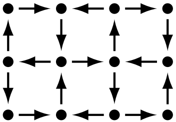

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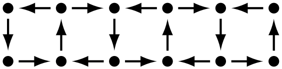

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Acyclic quivers

Definition

A quiver is acyclic if it does not contain directed cycles.

Theorem [Caldero, Keller 2006]

Acyclic quivers are mutation equivalent if and only if they are related via sink-source mutations.

Corollary

Orientations of non-isomorphic trees are not mutation equivalent.

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Hereditary properties

Definition

A property of quivers is hereditary if it descends from a quiver to its full subquiver (i.e., an induced directed subgraph).

Examples: quivers of finite type, quivers of finite mutation type.

Definition

A quiver is mutation-acyclic if it is mutation equivalent to an acyclic quiver.

Theorem [Buan, Marsh, Reiten 2008]

Mutation-acyclicity is a hereditary property.

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Find an elementary proof of this theorem (not using categorification).

Hereditary properties

Definition

A property of quivers is hereditary if it descends from a quiver to its full subquiver (i.e., an induced directed subgraph).

Examples: quivers of finite type, quivers of finite mutation type.

Definition

A quiver is mutation-acyclic if it is mutation equivalent to an acyclic quiver.

Theorem [Buan, Marsh, Reiten 2008]

Mutation-acyclicity is a hereditary property.

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Embeddability

Definition

We say that a quiver Q is embeddable into another quiver Q' if Q is a full subquiver of some quiver Q'' mutation equivalent to Q'.

It is more natural to talk about embeddability of mutation classes. Embeddability gives a partial order on mutation classes.

Theorem [SF, Zelevinsky; Felikson, Shapiro, Tumarkin]

Let Q' be a quiver on ≥ 3 vertices. Then

- some 2-vertex quiver Q with ≥ 2 arrows is embeddable into Q' if and only if Q' is a quiver of infinite type;
- some 2-vertex quiver Q with ≥ 3 arrows is embeddable into Q' if and only if Q' is of infinite mutation type.

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Detecting embeddability

Problem

For given Q and Q', decide whether Q is embeddable into Q'.

This is wide open for every fixed quiver Q on ≥ 2 vertices.
In particular, for any $k \geq 0$, the following problem is open.

Problem

Find an algorithm that detects whether a given quiver can be mutated to a quiver in which some two vertices are connected by exactly k arrows.

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More open problems on embeddability

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Let T and T' be two oriented trees (i.e., two orientations of trees).
True or false: T is embeddable into T' if and only if T' can be contracted to T.

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Find an elementary proof of the following result [Muller 2016]:
The Markov quiver is not embeddable into any grid quiver.

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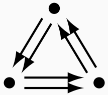

Plabic graphs and reddening sequences

In fact, the Markov quiver is not embeddable into the quiver of any reduced plabic graph. This follows from the following results:

Theorem [Muller 2016]

Existence of a reddening sequence is hereditary and mutation invariant.

The Markov quiver has no reddening sequence.

Theorem [Ford, Serhiyenko 2018]

The quiver of any reduced plabic graph has a reddening sequence.

Contrast this with:

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Any quiver is embeddable into the quiver of some plabic graph.

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Mutation cycles

Definition

A mutation cycle is a sequence of mutations, with no two consecutive mutations applied at the same vertex, that transforms a quiver Q into a quiver isomorphic to Q.

Motivation: linearizability/integrability of cluster dynamical systems.

Examples

If two vertices u and v in a quiver Q are connected by a single arrow, then the quiver $\mu_u \circ \mu_v \circ \mu_u \circ \mu_v \circ \mu_u(Q)$ is isomorphic to Q.

Additional examples include mutation cycles associated with:

- quivers of finite mutation type (cf. Dehn twists on surfaces);
- Zamolodchikov periodicity phenomena (DT transformations);
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Long mutation cycles

Let $\mathcal{C}$ be a mutation cycle passing through ℓ distinct n-vertex quivers.

In all aforementioned examples, the length ℓ of such mutation cycle $\mathcal{C}$ is bounded from above by a function of n.

Theorem [SF, Neville 2022+]

For any $n \geq 4$ and any N, there exists a mutation cycle $\mathcal{C}$ of length $\ell > N$ passing through ℓ distinct n-vertex quivers.

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Classify primitive mutation cycles. Short of that, catalog as many of them as possible.

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