Positroid varieties

Pavel Galashin (UCLA)

Open Problems in Algebraic Combinatorics 2022
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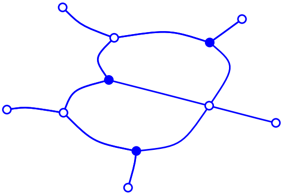
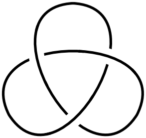
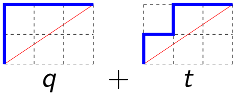
Joint work with Thomas Lam







The recipe for success

Step 1. Choose a variety

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^k \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

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[Deligne splitting / weight filtration $\to$ canonical second grading on $H^*(X)$]

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Question: Which variety should we choose?

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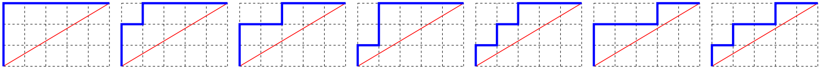

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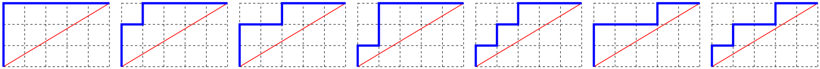

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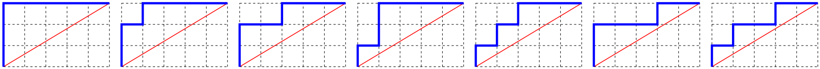

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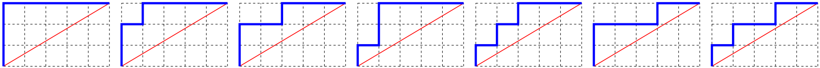

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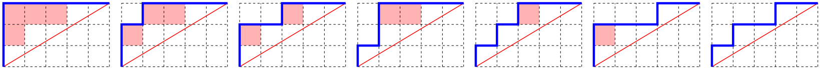

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- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let

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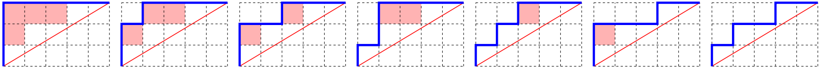
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

Example: $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



$$C''_{k,n-k}(t) = t^4 + t^3 + t^2 + t^2 + t^1 + t^1 + t^0.$$

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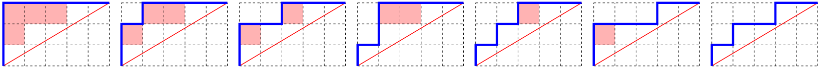

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For arbitrary $k \leqslant n$, the top open positroid variety is given by

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Recall: $\#X^\circ_{k,n}(\mathbb{F}_q) = C'_{k,n-k}(q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$ and $\#\,\mathrm{Gr}(k,n;\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q.$

Corollary

Let $\gcd(k,n) = 1$. *Then a uniformly random point of* $\mathrm{Gr}(k,n;\mathbb{F}_q)$ *belongs to* $\Pi^\circ_{k,n}(\mathbb{F}_q)$ *with probability* $\dfrac{(q-1)^n}{q^n - 1}$. $\longleftarrow$ *does not depend on* k*?!*

Open Problem

Prove this directly (without using knot theory).

Arbitrary positroid varieties

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- Point count? Poincaré polynomial? $\mathcal{P}(\Pi_f^\circ; q, t) = ?$

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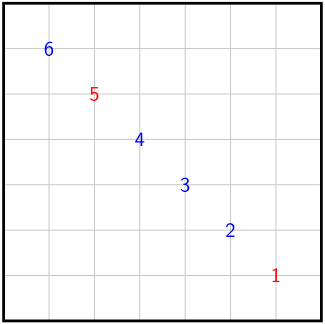

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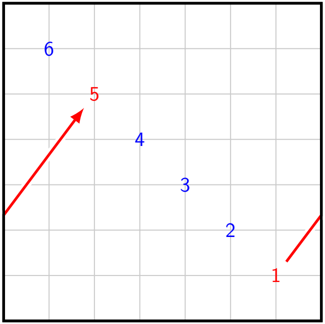

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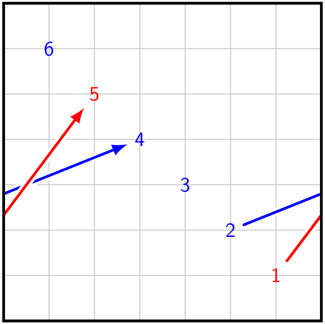

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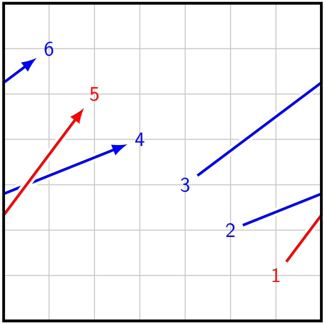

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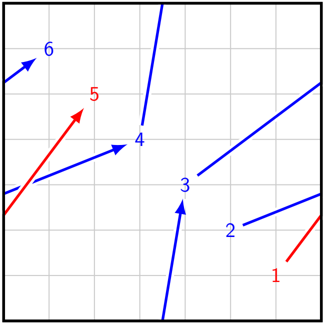

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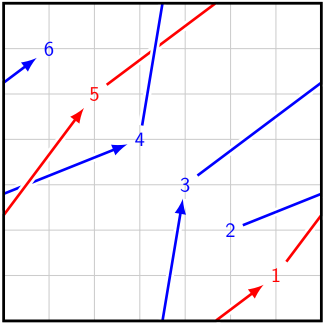

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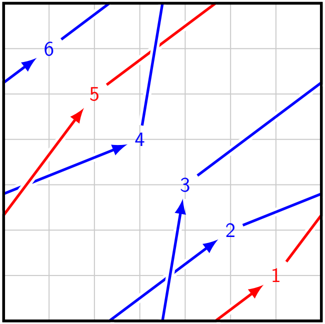

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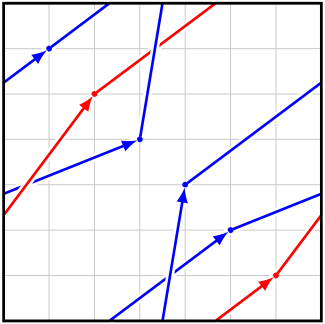

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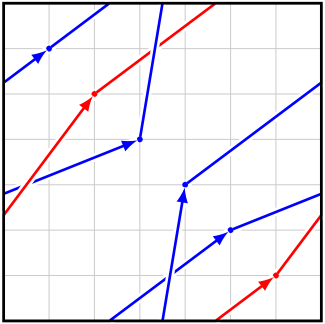

This construction: [G.–Lam '22+]. Related constructions: [G.–Lam '20], [Shende–Treumann–Williams–Zaslow '15], [Fomin–Pylyavskyy–Shustin–Thurston '17], [Casals–Gorsky–Gorsky–Simental '21]

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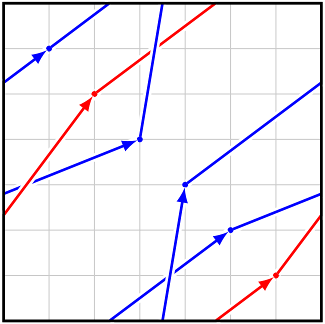

Conclusion

For each permutation $f \in S_n$, get a variety Π_f° and a link $L_f \subseteq \mathbb{R}^3$.

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Let $f \in S_n$. Then the point count of Π_f° is given by

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Arbitrary $f \in S_n$: LHS $=$ T-equivariant cohomology of Π_f° with compact support.

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A: No. Smallest known counterexample is a single cycle in S_{14}.

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\end{tikzcd}$$

Question

Is it true that we always have $\mathcal{P}(\Pi_f^\circ/T; q,t) \in \mathbb{N}[q,t]$?

<u>A:</u> No. Smallest known counterexample is a single cycle in S_{14}.

Problem

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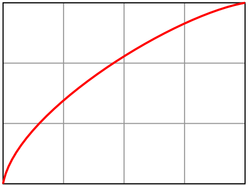

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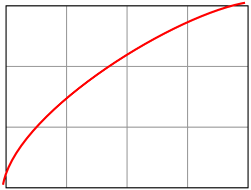

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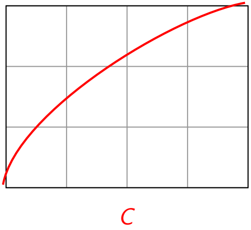

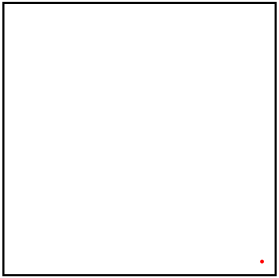

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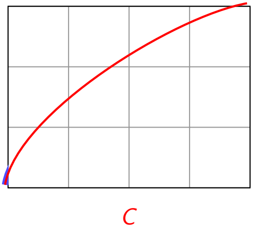

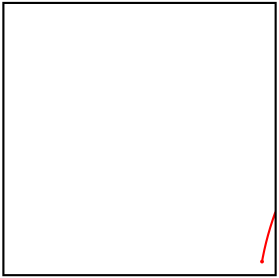

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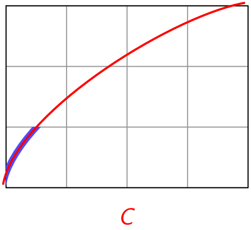

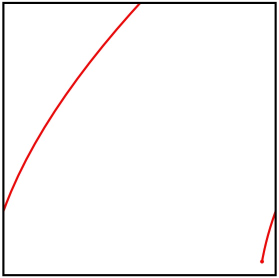

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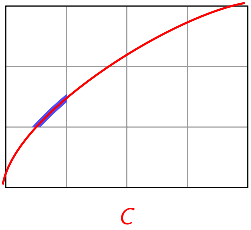

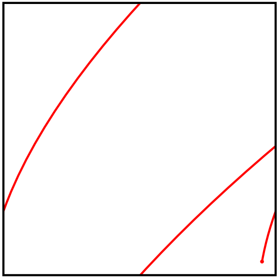

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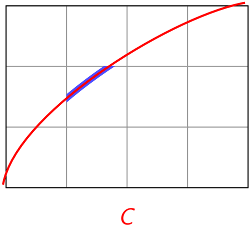

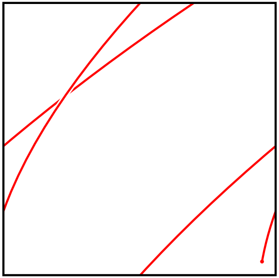

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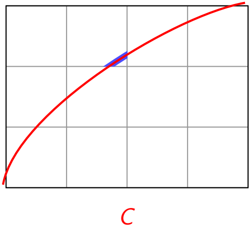

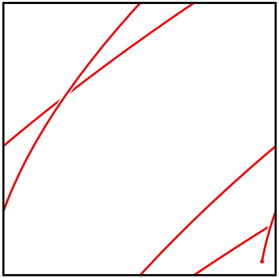

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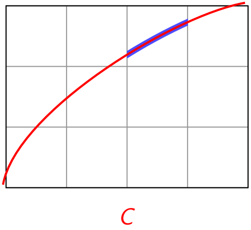

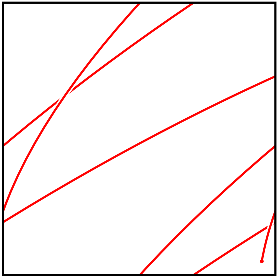

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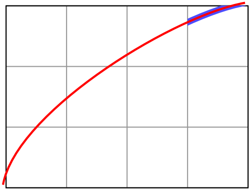

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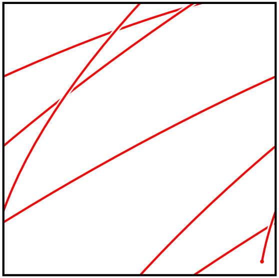

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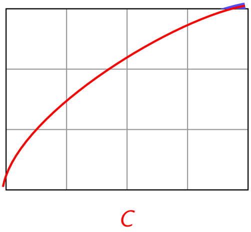

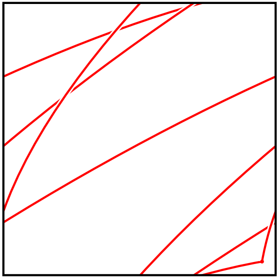

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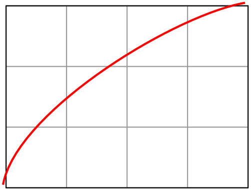

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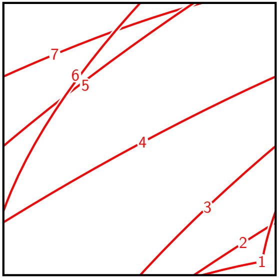

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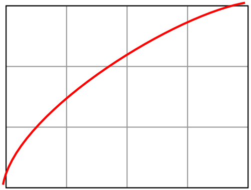

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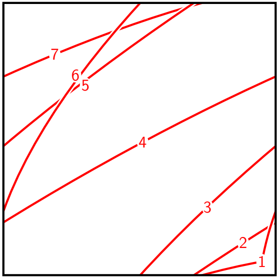

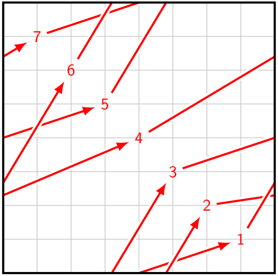

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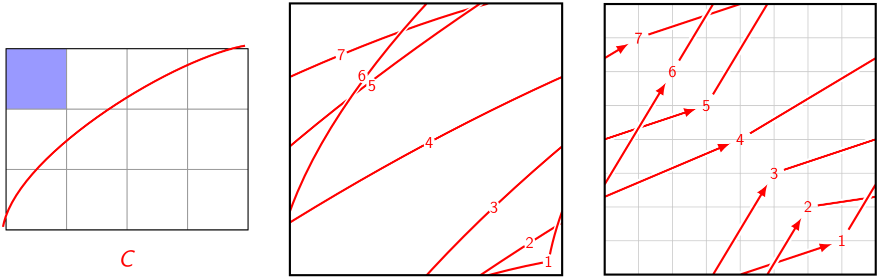

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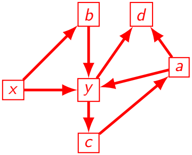

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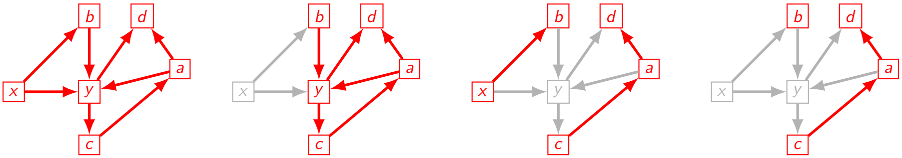

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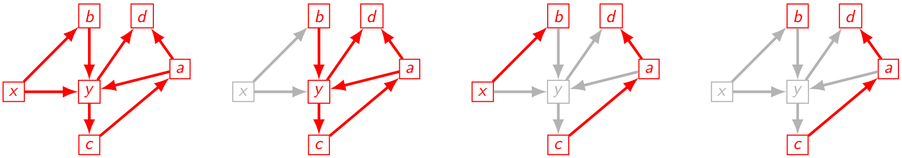

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Theorem (G.–Lam (2019) + Muller–Speyer (2016))

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Bonus: cluster algebras

Quiver $Q \longrightarrow$ cluster algebra $\mathcal{A}(Q)$. [Fomin–Zelevinsky '02]

Definition (Muller (2013))

- If Q has no arrows then it is locally acyclic.
- Suppose that after some mutations, we have found an arrow $x \to y$ in Q such that x has no incoming arrows from mutable vertices.*
- Suppose that $Q - x$, $Q - y$, and $Q - \{x, y\}$ are locally acyclic.
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Open Problem

Given a concave curve C, find a statistic dinv_C such that $\sum_{P \in \mathrm{Dyck}_C} t^{\mathrm{area}(P)} q^{\mathrm{dinv}_C(P)}$ is q,t-symmetric and q,t-unimodal.

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Generalization: links from plabic graphs

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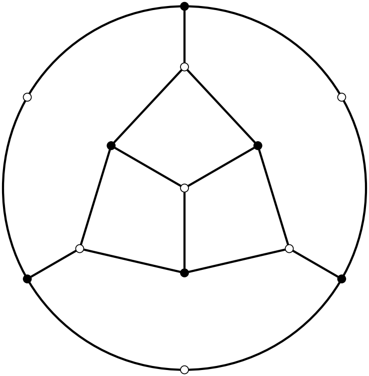

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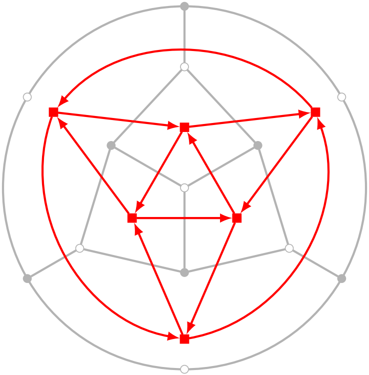

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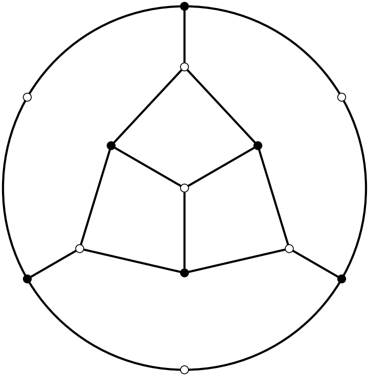

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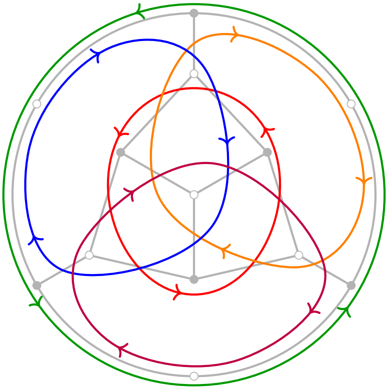

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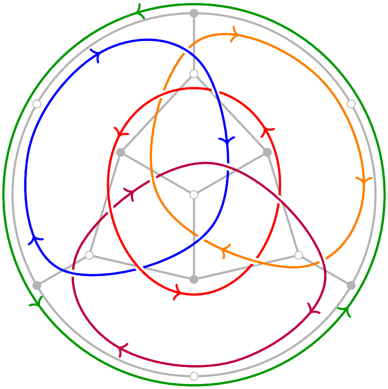

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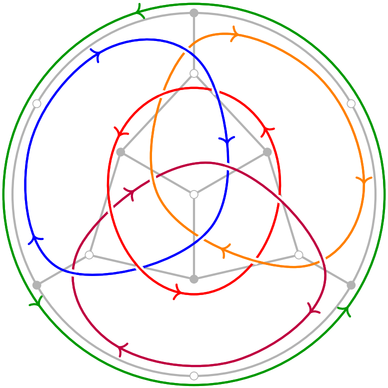

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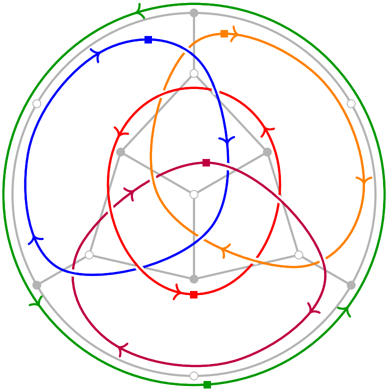

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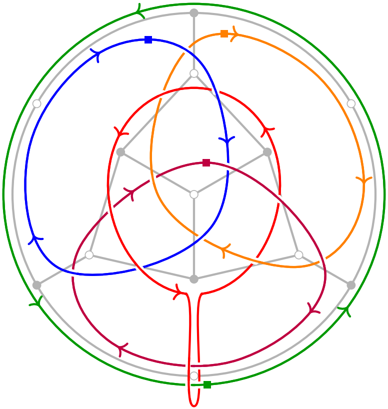

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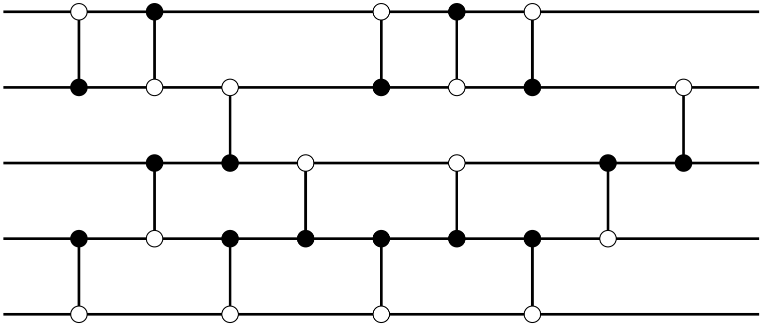

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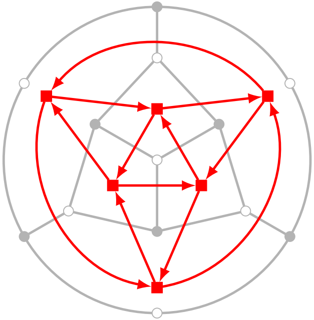

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