Open Problems in Dimers and Bundles

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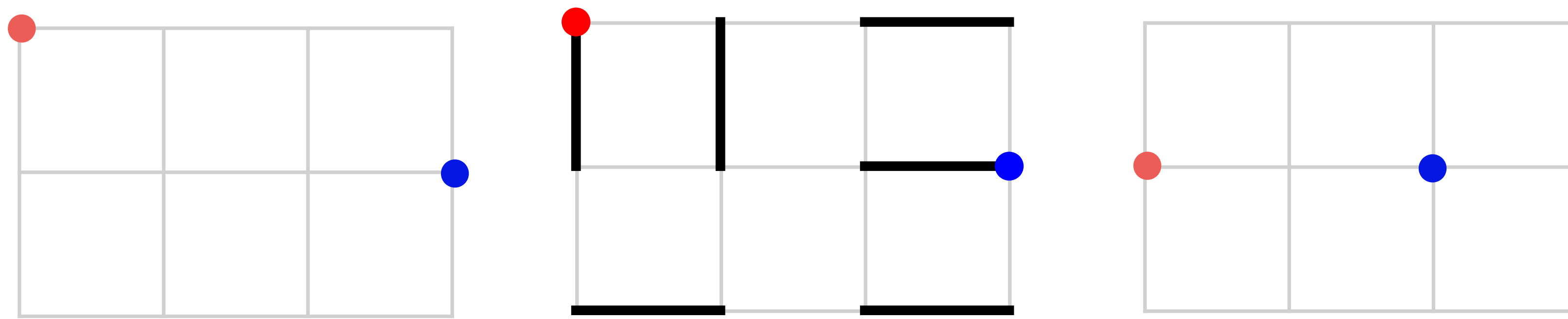
Problems

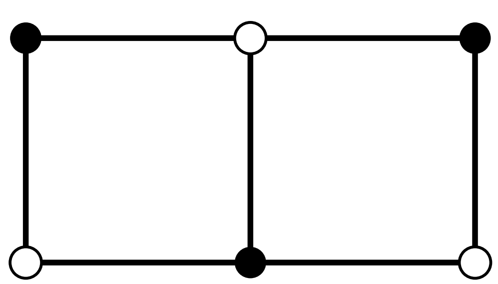
1. "Dimer random walks"
2. Weights $\leftrightarrow$ probabilities
3. Double-dimer loops
5. Double-dimer lamination coefficients
7. Triple-dimer web coefficients

Dimer random walks

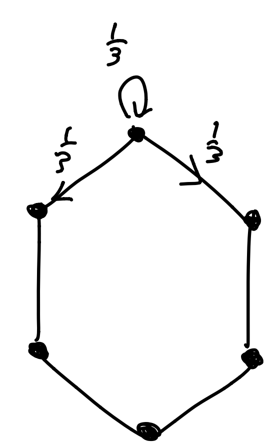


On a graph G, a (random) dimer cover is a (random) permutation of the vertices.

Problem 1. For an iid sequence of dimer covers, study the associated random walks on permutations.

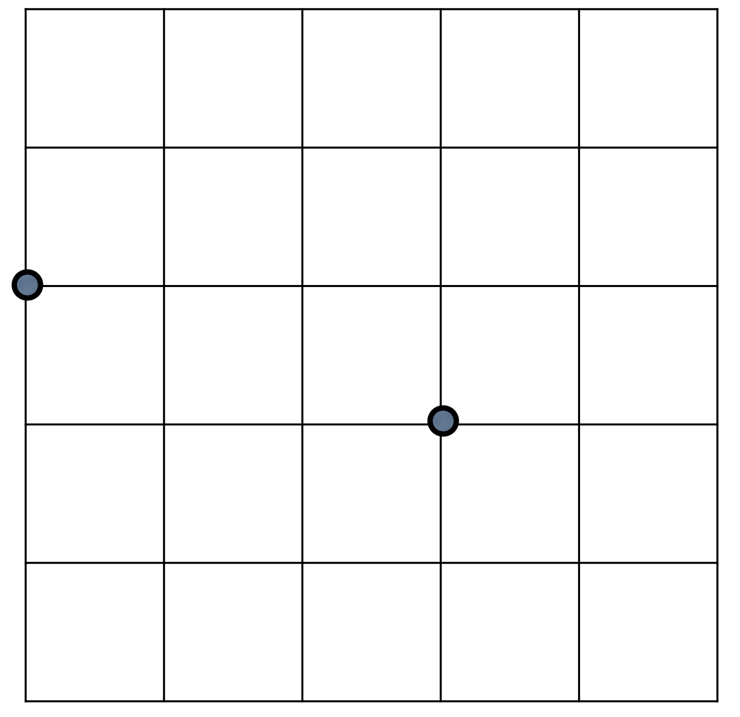


$$\frac{1}{3}(1 + e_{(12)} + e_{(2,3)})$$



For this graph, just record x coordinates of vertices, in S_3.

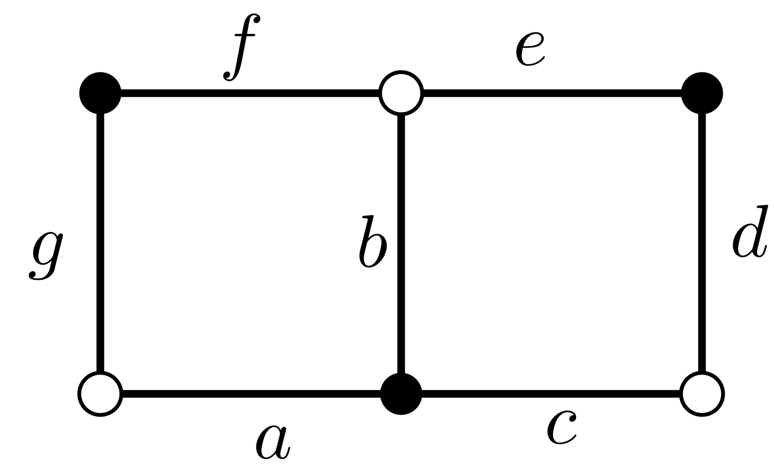
Ex. $n \times n$ grid on torus



Each particle does SRW, coupled to avoid each other.

Edge weights and bundles

Let ν_e be a positive real weight on each edge e.



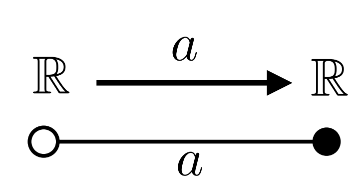
$$Pr(\text{dimer cover } m) = \frac{1}{Z} \prod_{e \in m} \nu_e$$

Observation: For bipartite graphs, we can think of ν_e as determining a connection on a line bundle.

Vector bundle: A copy V_u of a fixed vector space V at each vertex u.

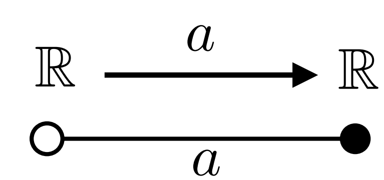
Connection: an isomorphism $\phi_{uv} : V_u \to V_v$ for adjacent vertices, with $\phi_{vu} = \phi_{uv}^{-1}$.

Line bundle: A vector bundle where V is one-dimensional.



For a line bundle, ϕ_{uv} is just multiplication by a (real) scalar.

For a bipartite graph with edge weights ν_e, define a line bundle with connection $\phi_{wb} = \nu_e$ on edges wb oriented from white to black.



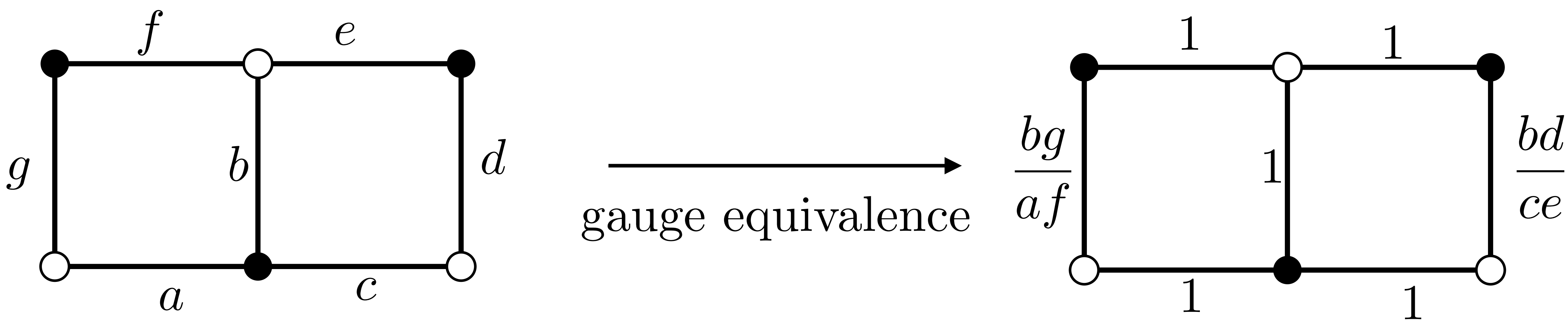
How is this helpful?

1. Gauge symmetry: changing basis at V_v corresponds to multiplying all edge weights (of edge incident to v) by a constant. This *does not change* the probability measure.

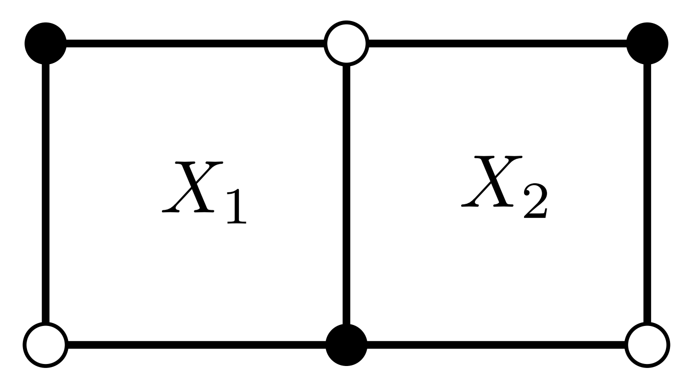
2. Generalizes naturally to other groups, e.g. $SL_n(\mathbb{R})$, see below.

3. Connects the problem to geometry

[K.-Sheffield '03], [Goncharov-K '13], [Lam-K-Ramassamy-Russkikh '19] [K.-'22]



$$\{\text{weights}\}/\{\text{gauge}\} \cong \mathbb{R}^F$$



$X_1 = \frac{af}{bg}$ $\qquad$ $X_2 = \frac{bd}{ce}$

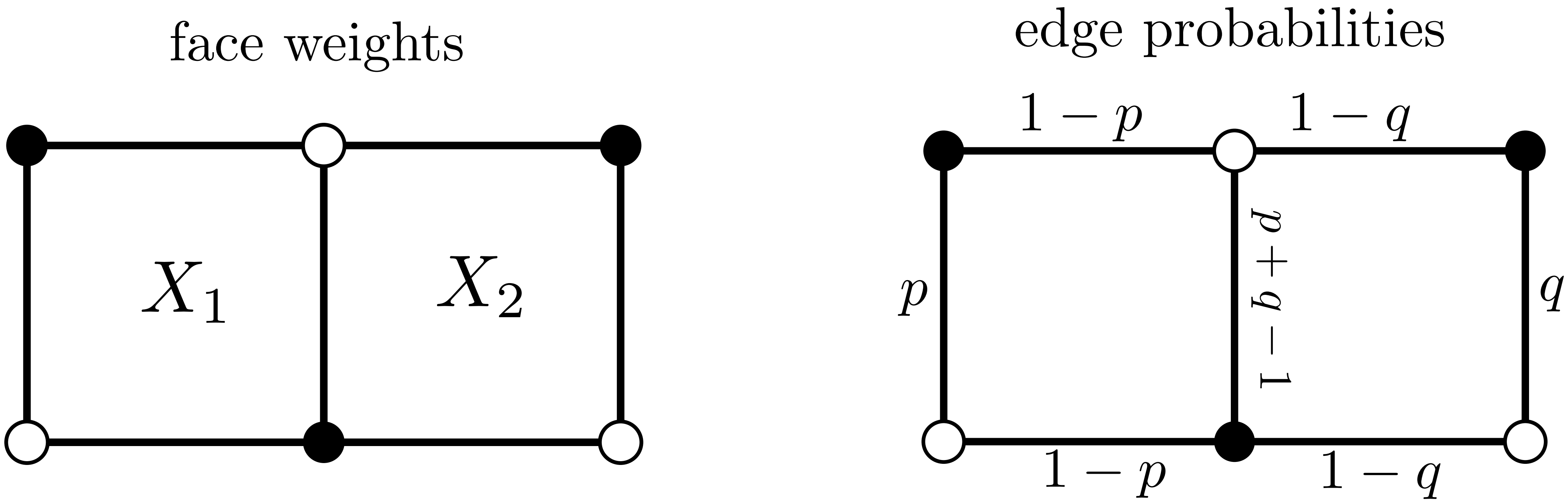
Problem 2. Study map from "face weights" $\{X_f\}$ to edge probabilities $\{Pr(e)\}$.
(cycle basis)

What is the space of edge probabilities?

$$\Omega = \{\text{Fractional dimer covers}\}$$
$$= \text{functions in } [0,1]^E \text{ summing to 1 at each vertex.}$$

Thm: $\Omega \subset [0,1]^E$ is a polytope whose vertices are the dimer covers.

Note the vector of edge probabilities $\vec{p} = (Pr(e))_{e \in E}$ lies in Ω; it is in fact the center of mass of the measure μ.



$$\Psi : \mathbb{R}_+^F \to \Omega$$

$$\Psi((X_f)_{f\in F}) = (Pr(e))_{e\in E}$$

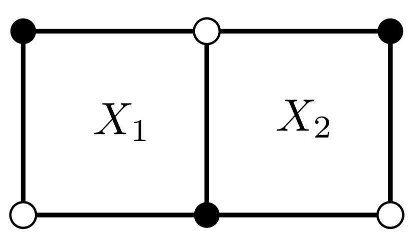
$$(X_1, X_2) \mapsto (p, q) = \left(\frac{1+X_2}{1+X_2+X_1X_2}, \frac{X_2+X_1X_2}{1+X_2+X_1X_2}\right)$$

Thm: If G is nondegenerate, Ψ is a diffeomorphism.

(Nondegenerate: each edge has $0 < Pr(e) < 1$)
(Nondegenerate: Ω has interior in $\mathbb{R}^F$.)

Problem 3. Is $\det \nabla \Psi$ subtraction free? Stable? What is the degree of Ψ as a rational map? Are all roots of $\Psi(\vec{X}) = \vec{p}$ real?

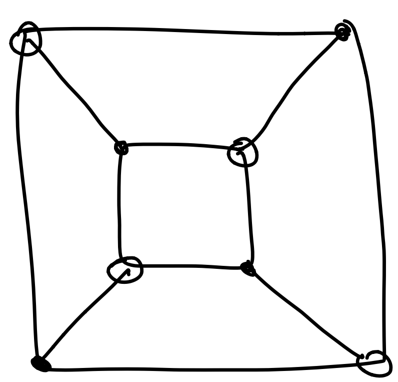
Example:



$$\det \nabla \Psi = \frac{X_2}{(1 + X_2 + X_1 X_2)^3}$$

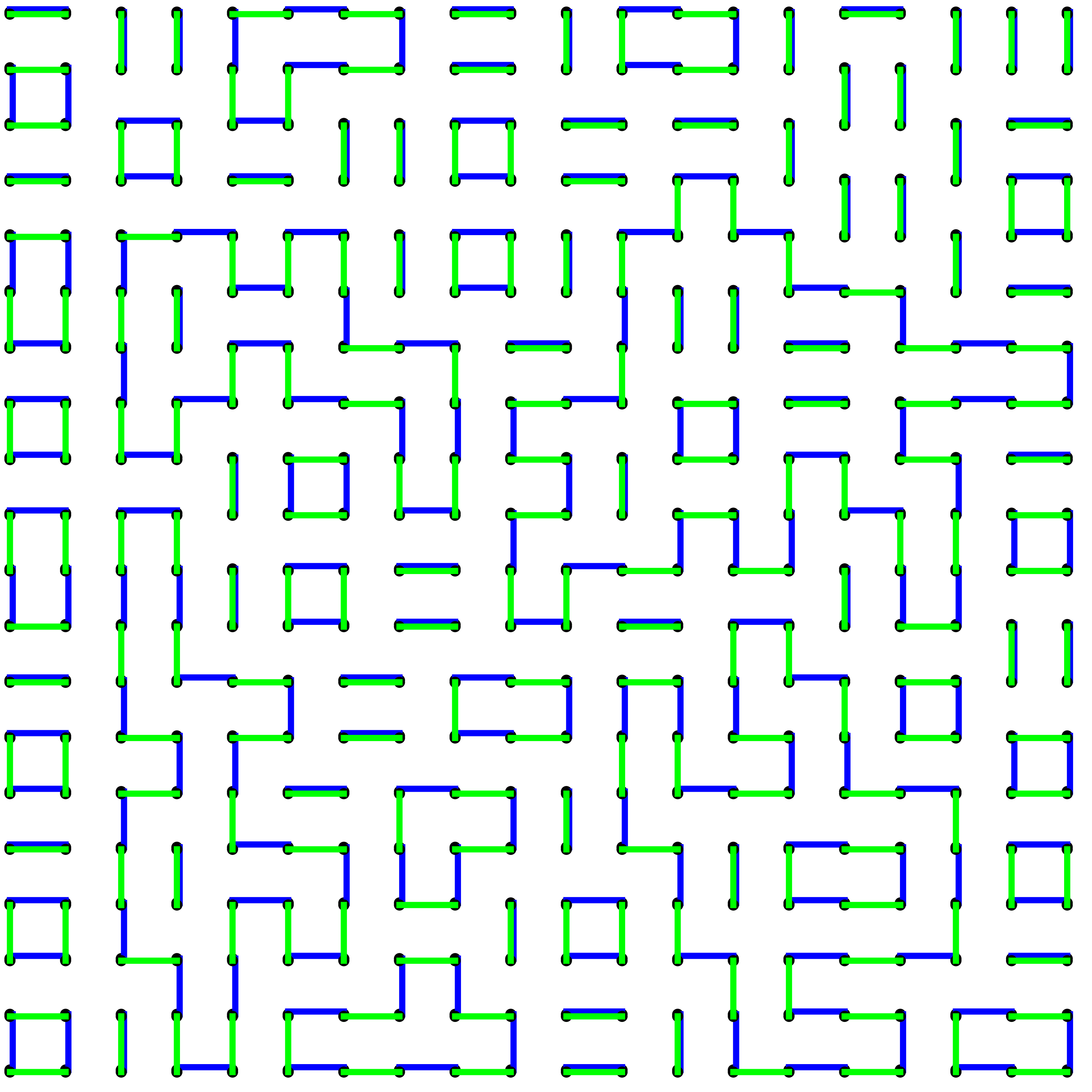
$$\Psi^{-1}(p,q) = \left(\frac{1-p}{p+q-1}, \frac{p+q-1}{1-q}\right)$$

Example:



Ψ has degree 2: Given $\vec{p}$, there are two choices of $\{X_f\}$ (only one of them positive) such that $\Psi(\{X_f\}) = \vec{p}$.

Double-dimer cover

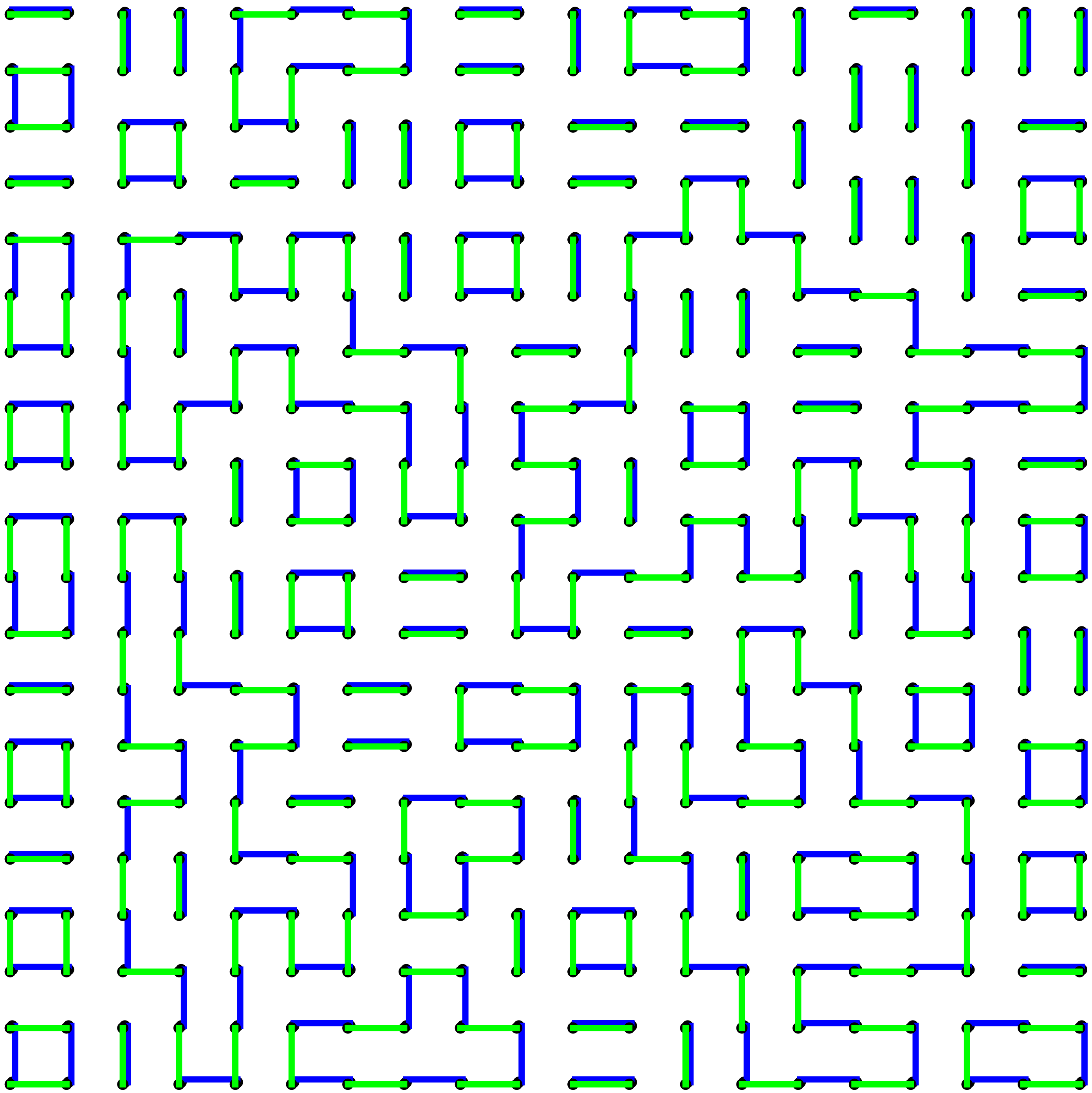


Ω_2 = set of double-dimer covers (forget colors)

Relating Ω_1 and Ω_2:

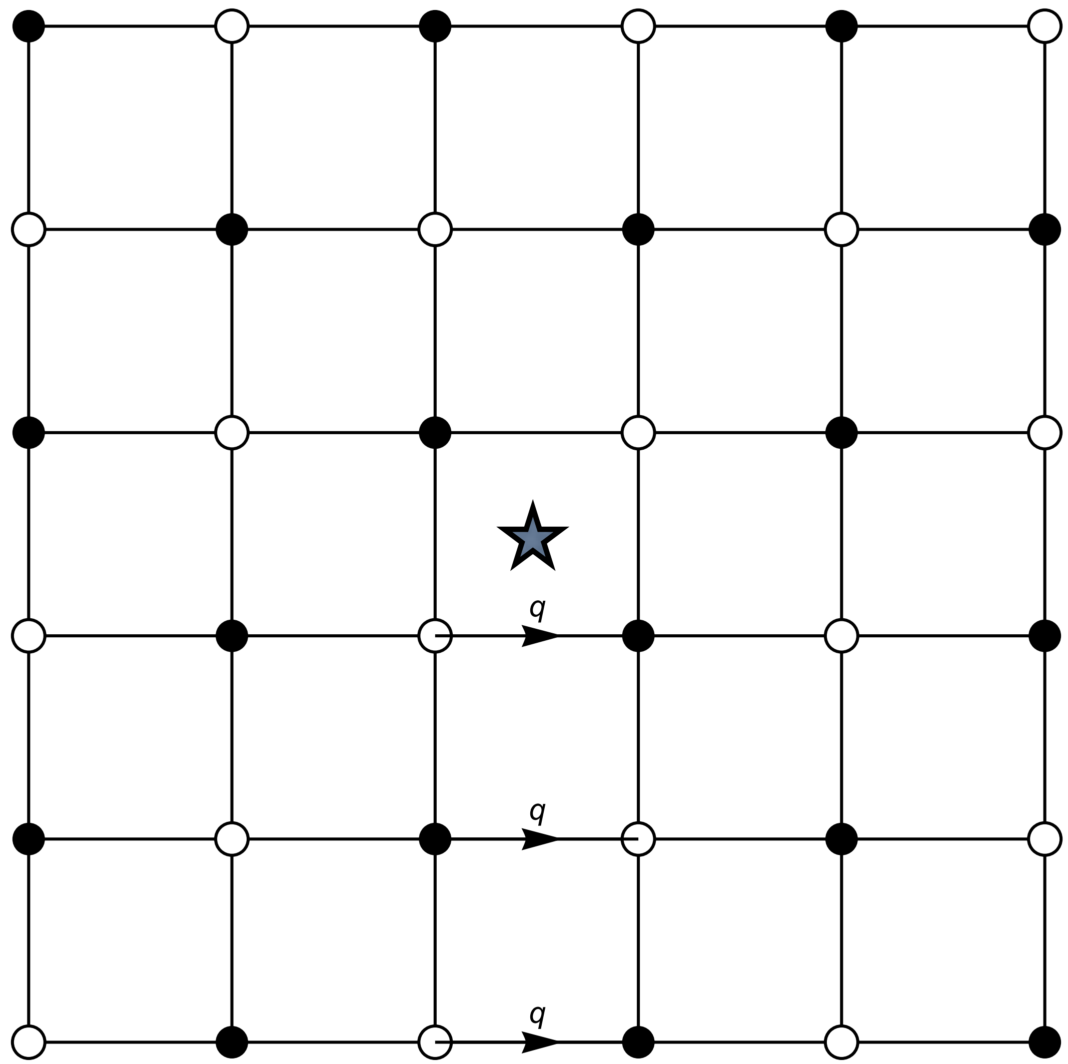
$$\sum_{m \in \Omega_2} 2^{\#\text{loops}} = |\Omega_1|^2$$

Problem 4. What is the distribution of loops (size, lengths, etc.) on $\mathbb{Z}^2$



$$Pr(\square) = \frac{1}{32} \qquad Pr(\ \text{▯}\ , \text{▭}\ ) = \frac{(\pi - 1)^2}{2\pi^4}$$

How to find the number of loops surrounding a point



$Z = \det K(q) \det K(1/q)$ "double dimer partition function"

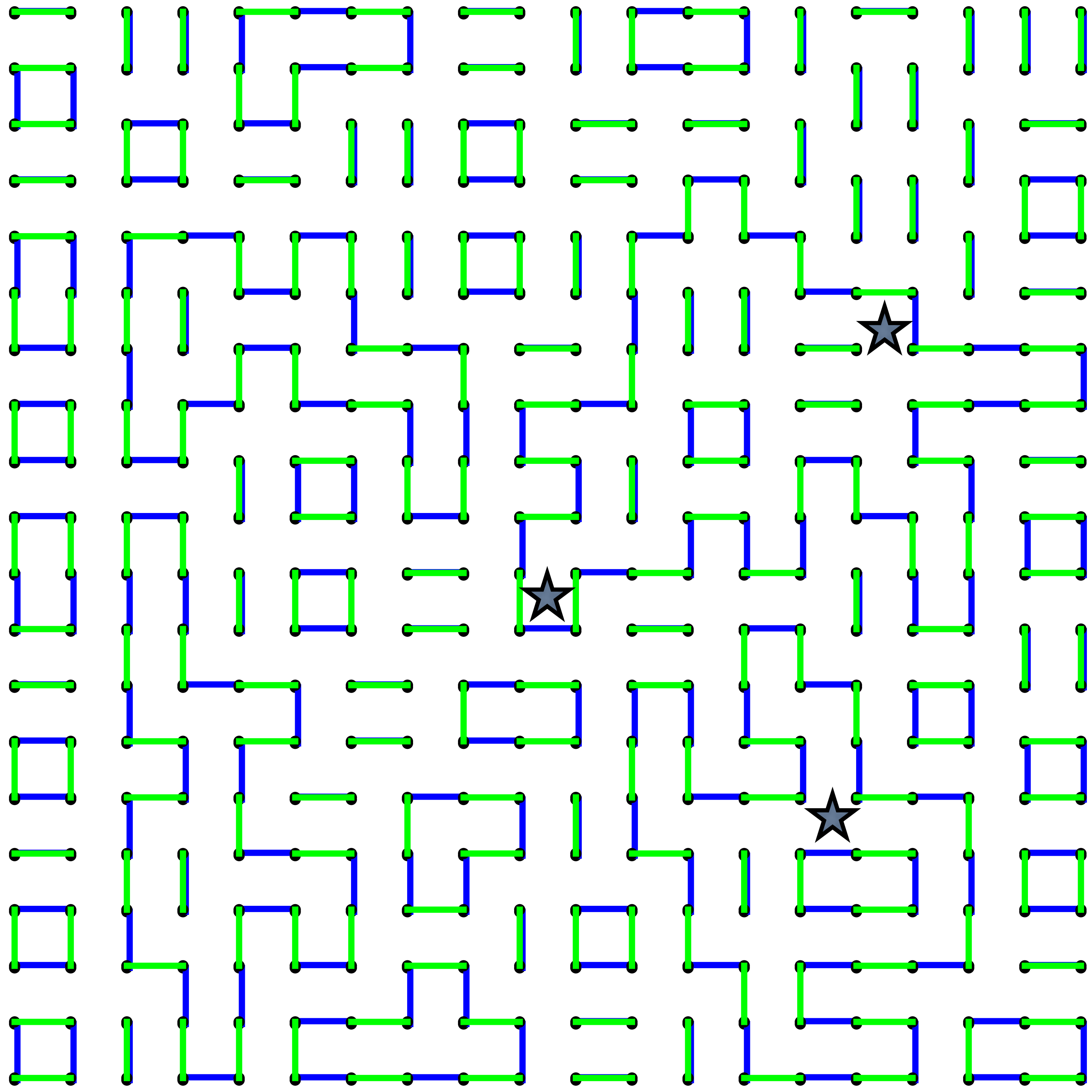
Each loop surrounding ★ contributes $q + 1/q$. (Other loops contribute 2).

$$Z(q) = \sum_{k \geq 0} C_k (q + 1/q)^k$$

where C_k counts configurations with k loops surrounding ★.

What about several points?

puncture some faces:

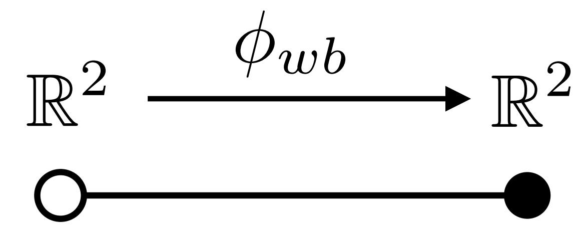


Problem 5. What is the probability that a loop in the double-dimer cover has a given homotopy class in Σ?

Thm [K.'14, Dubedat '19, Basok-Chelkak '20]: These probabilities are *conformally invariant* in the scaling limit.

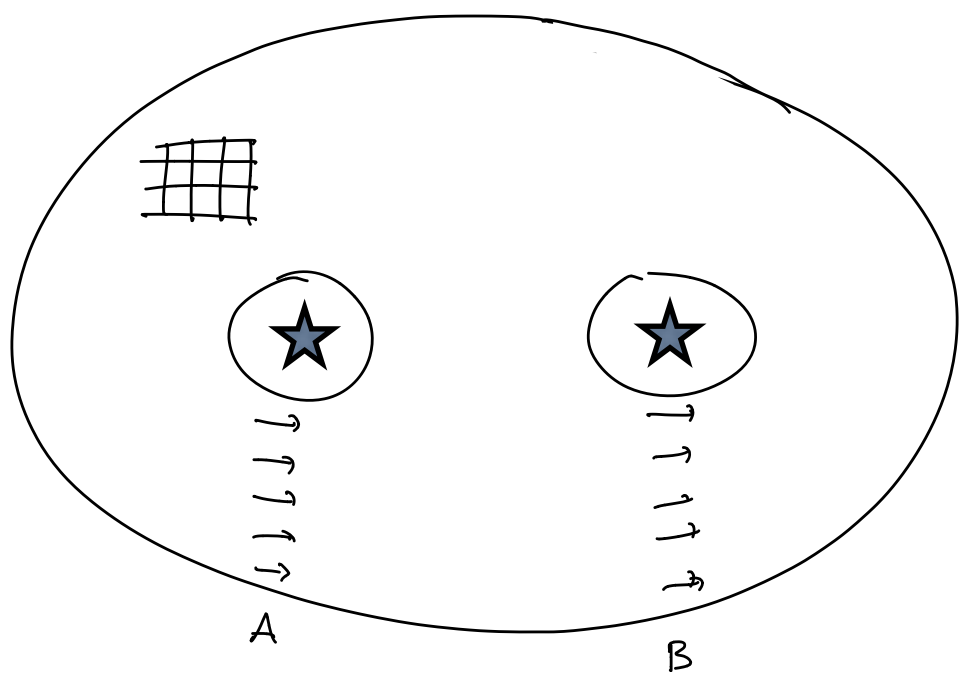
This type of question can be answered with the help of an SL_2-local system.

Let $\Phi = \{\phi_e\}_{e \in E}$ be an SL_2-connection on G:

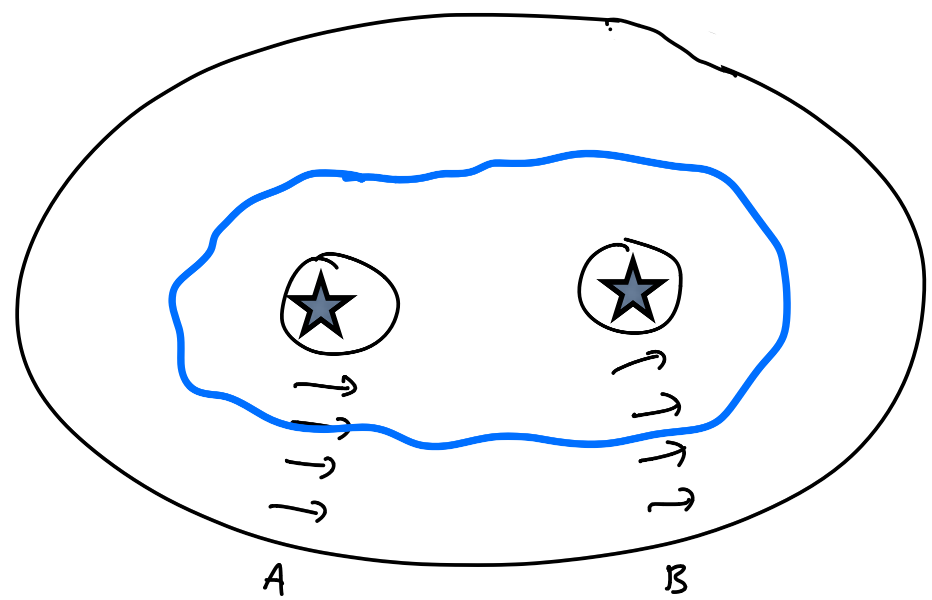


For a loop γ, let ϕ_γ be the monodromy of Φ around γ.

composition of ϕ's around γ



We assume Φ is *flat*: trivial monodromy around each contractible loop.



$$Tr(\phi_\gamma) = Tr(AB)$$

For a loop γ, let ϕ_γ be the monodromy of Φ around γ.

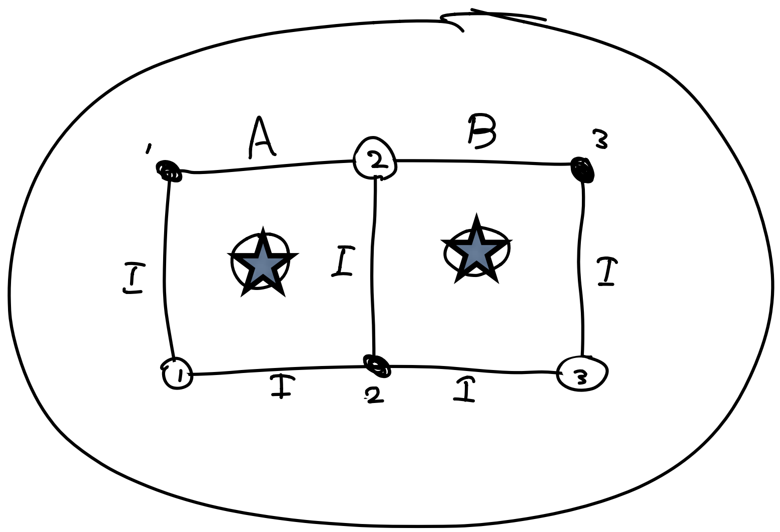
For any double-dimer configuration $m \in \Omega_2$ we define

$$Tr(m) = \prod_{\text{loops } \gamma \text{ of } m} Tr(\phi_\gamma)$$

in SL_2, trace does not depend on orientation.

The trace "detects" the homotopy type of the loops

We define a Kasteleyn matrix $K(\Phi)$ on G with an SL_2-local system Φ.



$$\begin{pmatrix} I & I & 0 \\ A & -I & B \\ 0 & I & I \end{pmatrix} \in M_3(SL_2(\mathbb{R}))$$

really $M_3(M_2(\mathbb{R}))$

Thm[K, 2016] $\det(\tilde{K}(\Phi)) = \sum_{m\in\Omega_2} Tr(m)$.

(remove "inner" parentheses)

For example: $\det \tilde{K}(I) = \sum_{m\in\Omega_2} 2^{\#\text{loops}}$

Thm[K, 2016] $\det(\tilde{K}(\Phi)) = \sum_{m \in \Omega_2} Tr(m)$.

(remove "inner" parentheses)

Rewrite this sum: $\det(\tilde{K}(\Phi)) = \sum_{\lambda \in \Lambda} C_\lambda \mathrm{Tr}(\lambda)$

where λ runs over isotopy classes of simple laminations.

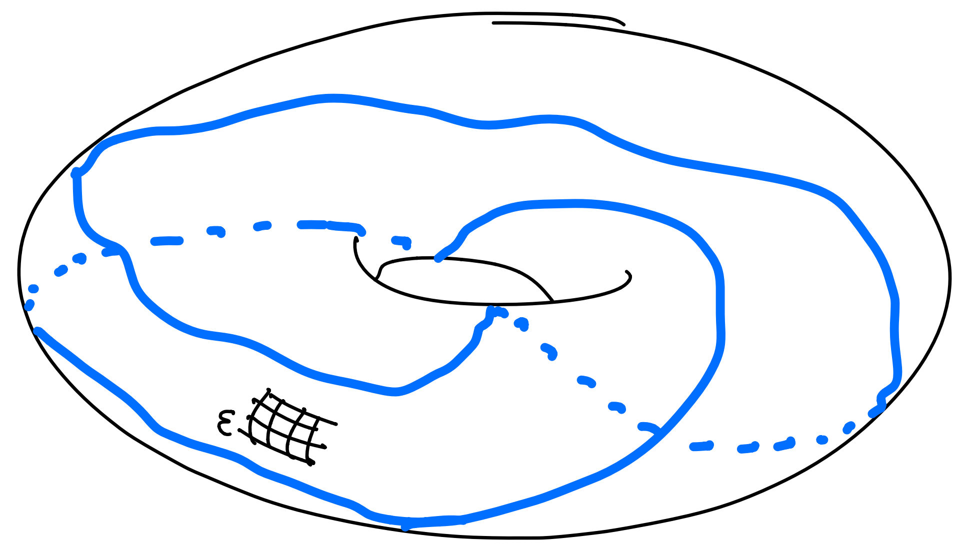
collections of disjoint simple closed curves

Thm[Fock-Goncharov '13]: Traces of simple laminations form a linear basis for regular functions on the character variety.

Cor: C_λ is determined by $K(\Phi)$.

Problem 6. How to extract C_λ from $K(\Phi)$?

Example: torus with a fine square grid and m a random double-dimer cover,



In the limit $\varepsilon \to 0$,

$Pr(m$ has k curves of homology class $(i,j)) \propto e^{-Q(ik,jk)}$

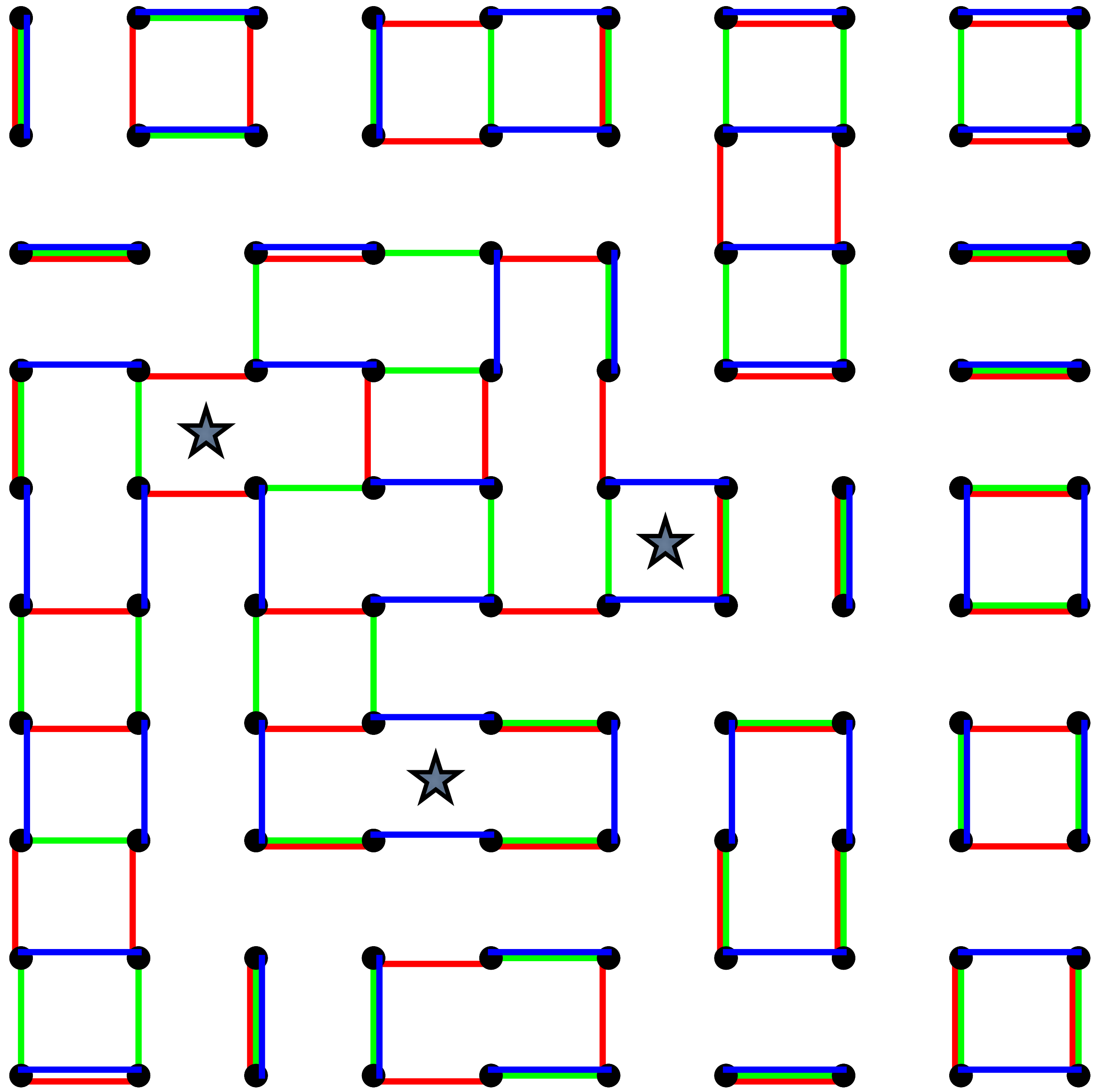
for a certain quadratic form Q.

[Boutillier-de Tilière '09]

The constant of proportionality *is a theta function...*

n-dimer model

Now, superpose n dimer covers (and forget the colors, but remember multiplicities)



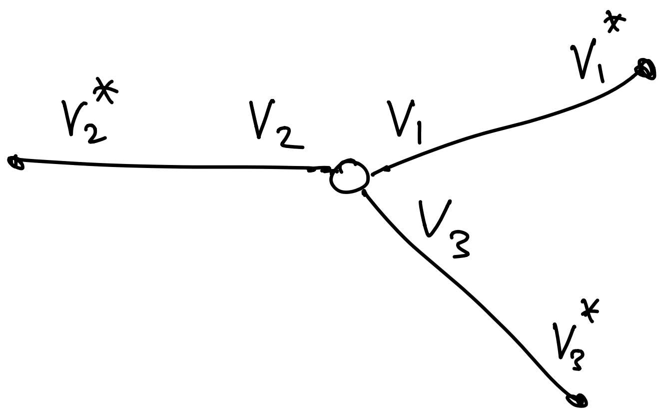
We get an "n-multiweb" or "n-fold dimer cover"

$$\Omega_n = \{n\text{-multiwebs}\}$$

Thm:[Douglas,K,Shi] For a planar graph G,

$$\pm\det(\tilde{K}(\Phi)) = \sum_{m\in\Omega_n(G)} Tr(m).$$

Trace of a web $(n = 3)$



$V_i \cong \mathbb{R}^3$ with basis e_1, e_2, e_3

$v_w \in V_1 \otimes V_2 \otimes V_3$

$$v_w = \sum_{\sigma \in S_3} (-1)^\sigma e^1_{\sigma(1)} \otimes e^2_{\sigma(2)} \otimes e^3_{\sigma(3)}$$

"codeterminant"

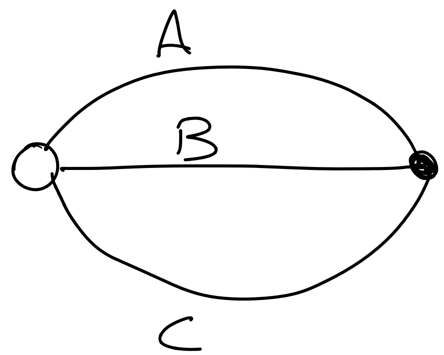
$v_b \in V_1^* \otimes V_2^* \otimes V_3^*$

$$v_b = \sum_{\sigma \in S_3} (-1)^\sigma f^1_{\sigma(1)} \otimes f^2_{\sigma(2)} \otimes f^3_{\sigma(3)}$$

invariant under SL_n-base change

$$Tr(m) = \left\langle \bigotimes_{w \in W} v_w \middle| \bigotimes_{e = wb} \phi_{wb} \middle| \bigotimes_{b \in B} v_b \right\rangle$$

3-web example



V basis e_r, e_g, e_b

$$v_w = e_r^1 \otimes e_g^2 \otimes e_b^3 - e_r^1 \otimes e_b^2 \otimes e_g^3 + \cdots - e_b^1 \otimes e_g^2 \otimes e_r^3$$

$A_{rr}B_{gg}C_{bb}$

$$v_b = f_r^1 \otimes f_g^2 \otimes f_b^3 - f_r^1 \otimes f_b^2 \otimes f_g^3 + \cdots - f_b^1 \otimes f_g^2 \otimes f_r^3$$

$$Tr(m) = A_{rr}B_{gg}C_{bb} + \cdots + A_{bb}B_{gg}C_{rr}$$

$$Tr(m) = Tr(AB^{-1})Tr(CB^{-1}) - Tr(AB^{-1}CB^{-1})$$

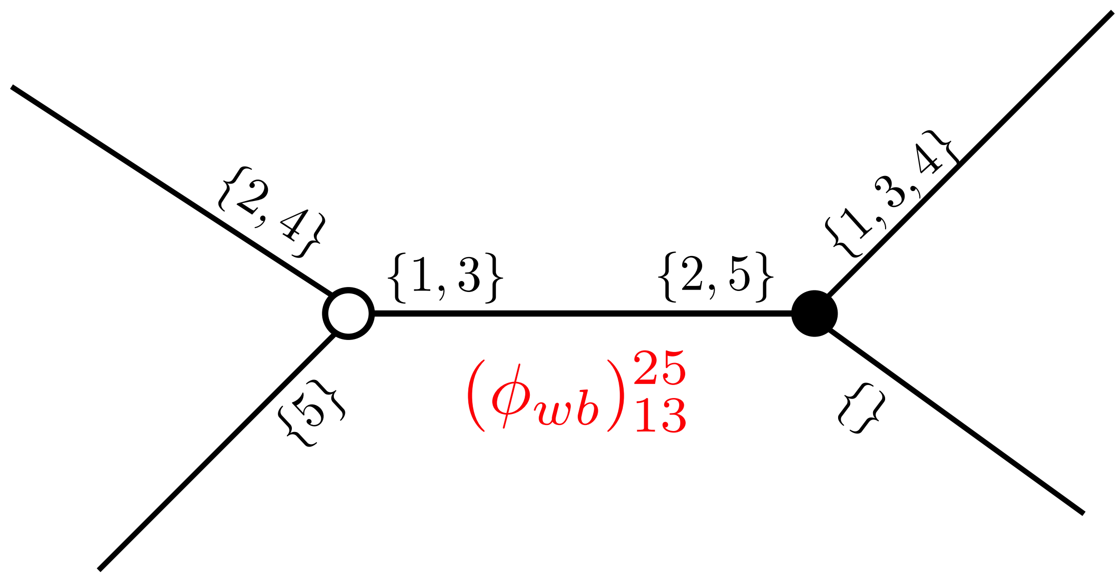
$$= [xyz]\,\det(xA + yB + zC)$$

“Coloring” definition of trace

Assign colors in $[n]$ to the half-edges at each vertex such that:

- An edge of multiplicity k gets two sets S_e, T_e of k colors.
- Sets at a vertex partition $[n]$.

$n = 5$



Assign to an edge wb with colors S_e, T_e the minor $(\phi_w)^{T_e}_{S_e}$.

Prop:
$$Tr(m) = \sum_{\text{colorings } c} (-1)^c \prod_e (\phi_e)^{T_e}_{S_e}$$
where $(-1)^c$ is the product of signatures at each vertex, depending on ordering of colors.

Thm:[Douglas,K,Shi] For a planar graph G,

$$\pm\det(\tilde{K}(\Phi)) = \sum_{m\in\Omega_n(G)} Tr(m).$$

Note: When n is odd, sign depends on an artificial ordering of vertices.

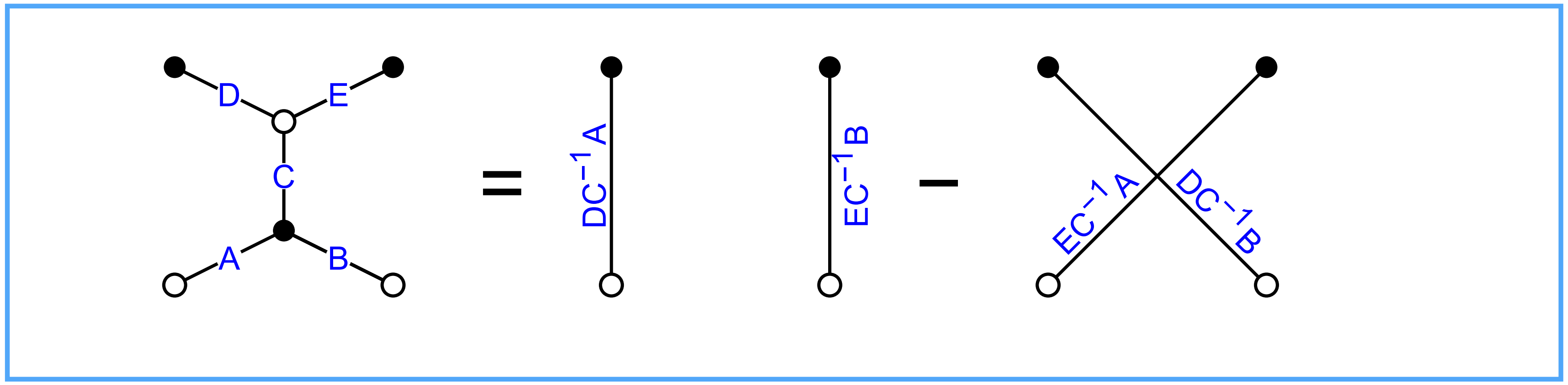
Note: When n is even, sign of trace depends on an artificial choice of linear ordering of edges at each vertex.

However typical multiwebs m are not *reduced*.

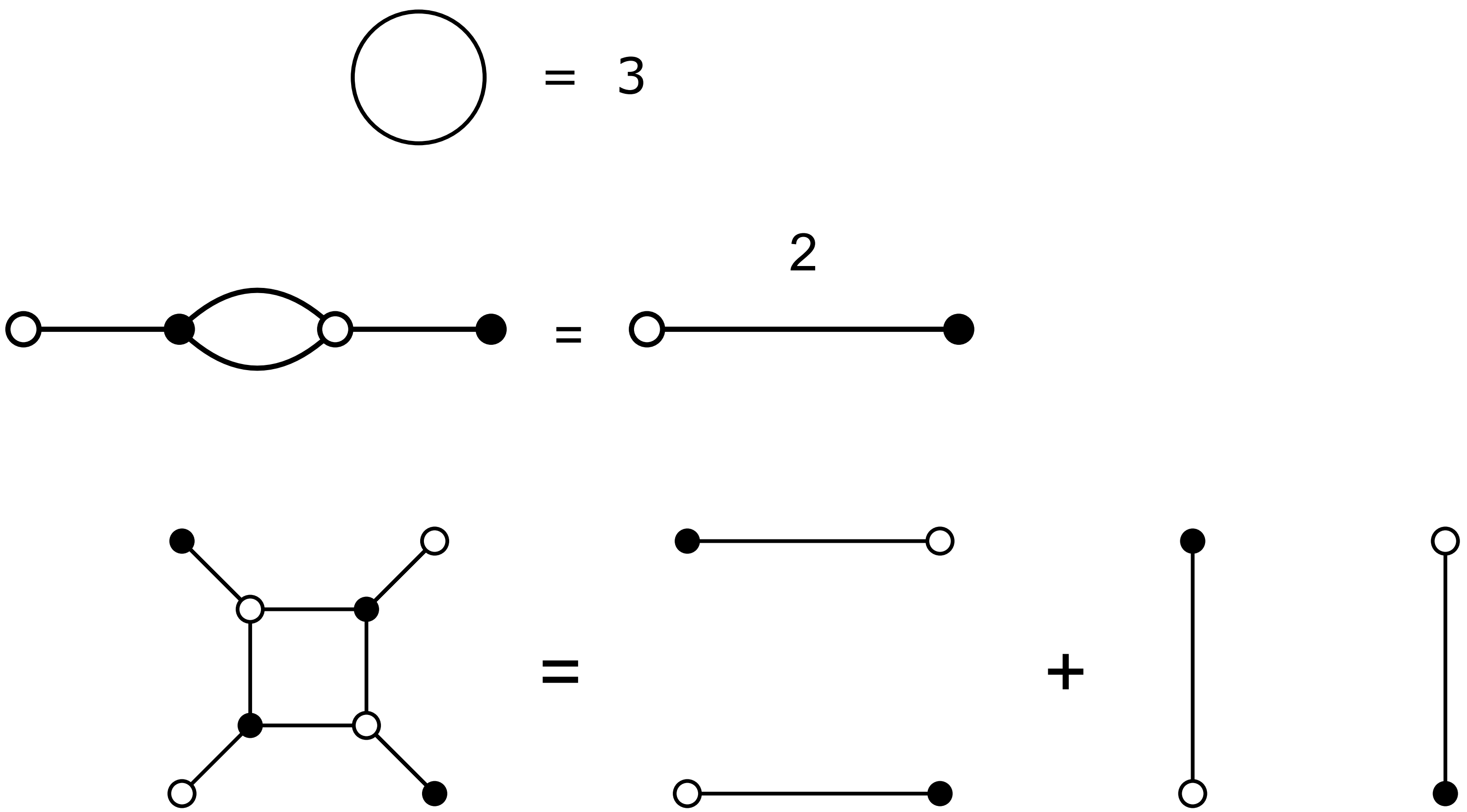
Thm[Sikora-Westbury] Traces of reduced (i.e. nonelliptic) webs form a basis for regular functions on the SL_3-character variety.

reduced = no topologically trivial faces of degree < 6.

Web reductions (skein relations) $n = 3$:



Basic skein relation



Reductions preserving planarity (consequences of the basic skein relation)

For a 3-multiweb m on a graph on a surface with a flat SL_3-connection

$$Tr(m) = \sum_{m'} Tr(m')$$

where the sum is over reduced (non-elliptic) webs m' in m. Even though the reduction is not canonical, the topological types of the m' are.

Consequently

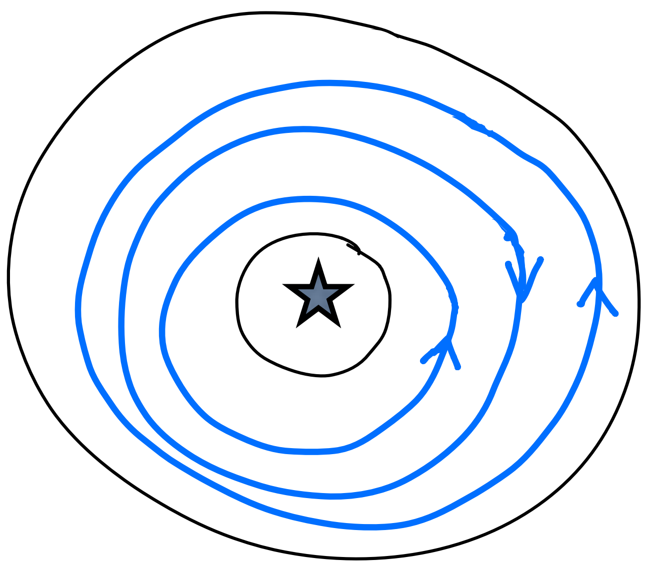
Thm: $\det(\tilde{K}(\Phi)) = \sum_{\lambda \in \Lambda_3} C_\lambda Tr(\lambda)$ where the C_λ are functions of $\det \tilde{K}(\Phi)$.

isotopy classes of reduced webs

Problem 7. How to extract C_λ?

Problem 8 Is there a "canonical" set of reduced 3-webs associated to a given 3-web?

Example. On an annulus, every reduced 3-multiweb is a union of topologically nontrivial oriented loops



$$\det \tilde{K}(A) = \sum_{i,j\geq 0} C_{i,j} Tr(A)^i Tr(A^{-1})^j$$

Prop: For an $n \times m$ grid on a cylinder, as $n, m \to \infty$ with $n/m \to \tau$,

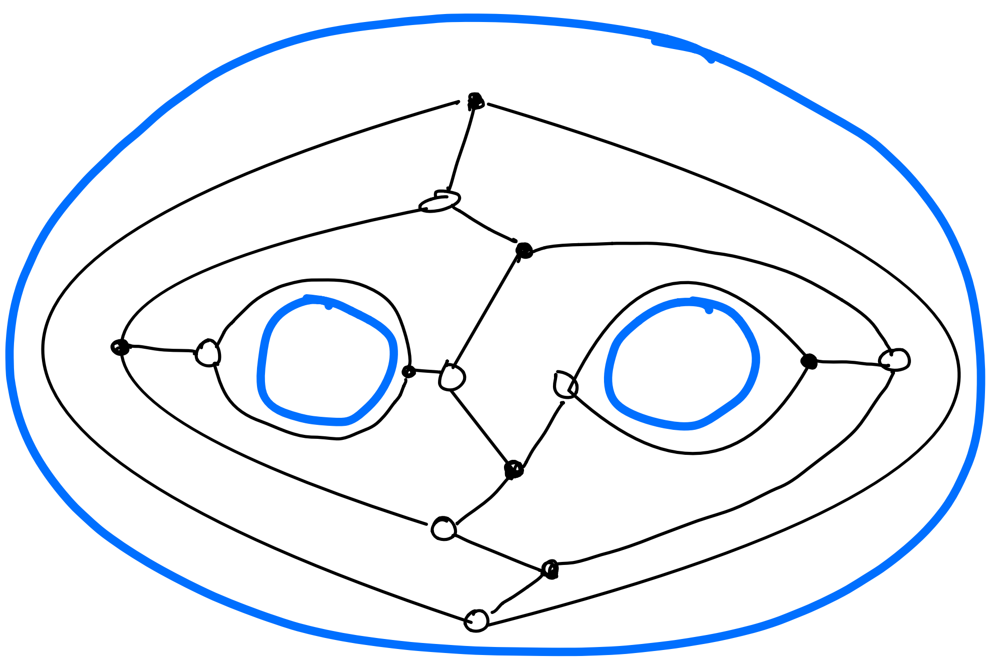
$$\sum_{i,j\geq 0} C_{i,j} u^i v^j = C' \prod_{j=1}^{\infty} (1 + uq^j + vq^{2j} + q^{3j})(1 + vq^j + uq^{2j} + q^{3j})$$

$u = Tr(A), v = Tr(A^{-1})$

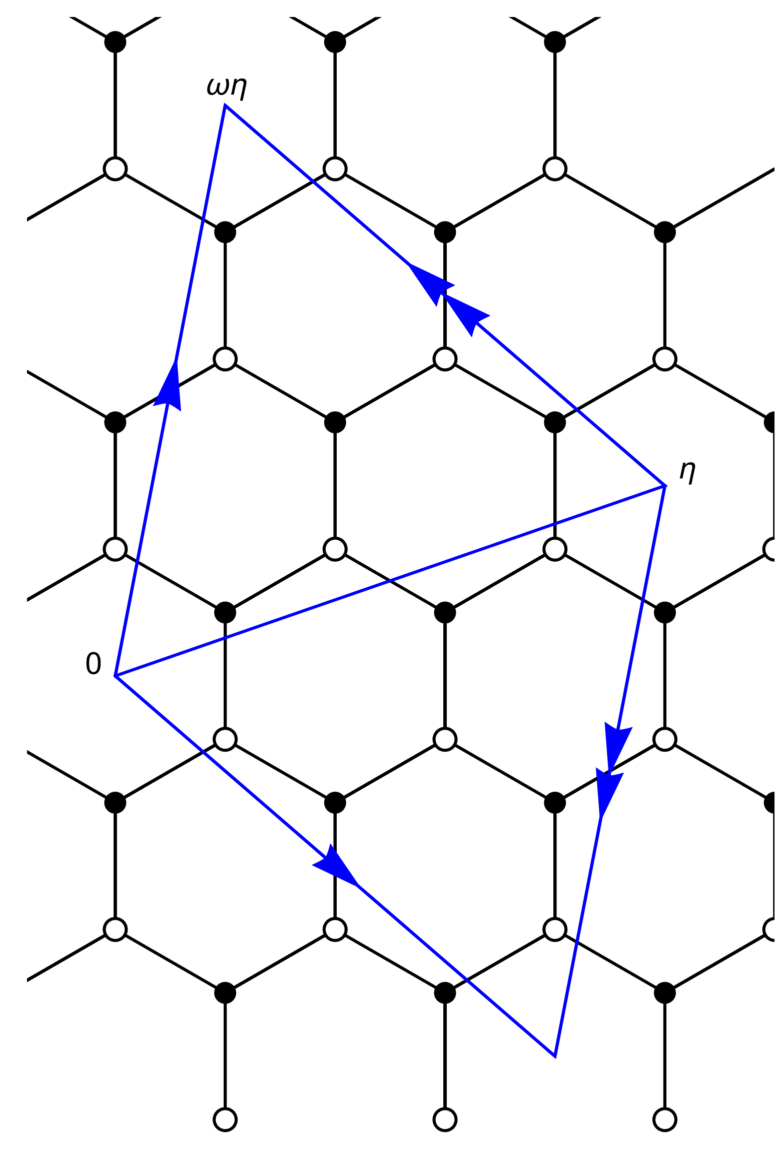
where $q = e^{-\pi n/m}$.

Example. On a pair of pants, every reduced 3-multiweb is a union of topologically nontrivial oriented loops and possibly one W_η component

$$\eta = a + be^{\pi i/3}, \quad a, b \in \mathbb{Z}_+$$

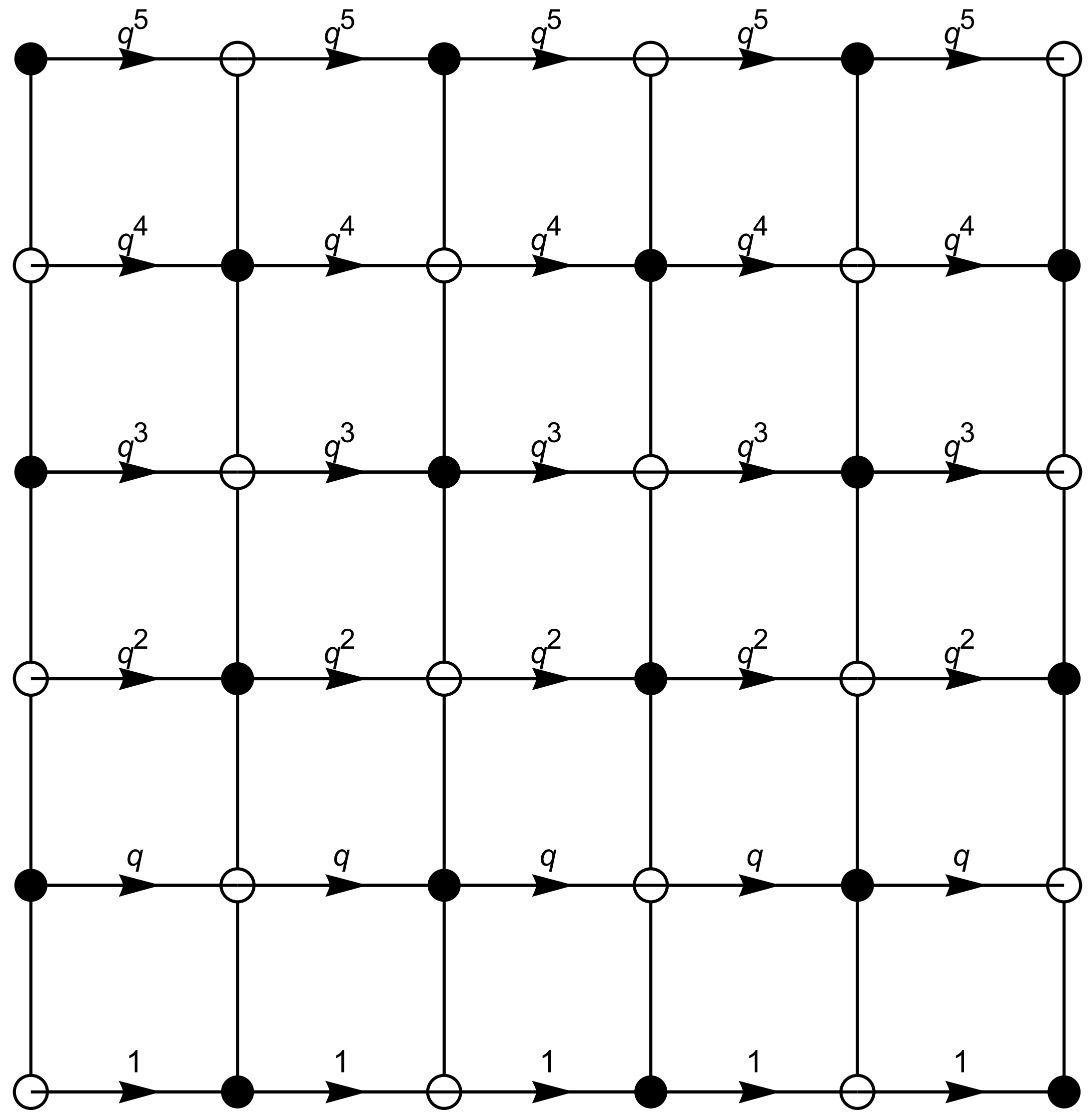


$W_{2+\omega}$



THANK YOU

Magnetic dimer model



We put a line bundle with monodromy q around every face.

Each loop contributes weight $q^A + q^{-A}$

$$\det K(q) \det K(1/q) = \sum_{\text{dd covers}} \prod_{\text{loops } \gamma} (q^{A(\gamma)} + q^{-A(\gamma)})$$