Combinatorial expressions for the nabla operator

Anton Mellit*

University of Vienna

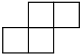
OPAC, University of Minnesota
May 2022

*based on joint work with Erik Carlsson

Motivating example

Fix a skew shape $\lambda \setminus \mu$. Ex.:

$$\lambda \setminus \mu = (3,2) \setminus (1,0) =$$



Fix content $c = 1^{\nu_1} 2^{\nu_2} \cdots$, i.e. a multiset of integers. Ex.:

$$c = (1,1,2,2)$$

Count semistandard Young tableaux of shape $\lambda \setminus \mu$ and content c:

	1	1
2	2	

	1	2
1	2	

Motivating example

shape $\rightsquigarrow$ Schur function $s_{\lambda\setminus\mu}$ content $\rightsquigarrow$ $h_\nu := \prod_i h_{\nu_i}$

The answer is:

$$(s_{\lambda\setminus\mu}, h_\nu)$$

$$s_{(3,2)\setminus(1,0)} = s_{(2,2)} + s_{(3,1)}$$

$$h_{(2,2)} = s_{(4)} + s_{(3,1)} + s_{(2,2)}$$

$$(s_{(3,2)\setminus(1,0)}, h_{(2,2)}) = 2$$

Motivating example

shape $\rightsquigarrow$ Schur function $s_{\lambda\backslash\mu}$ content $\rightsquigarrow$ $h_\nu := \prod_i h_{\nu_i}$

The answer is:

$$(s_{\lambda\backslash\mu}, h_\nu)$$

$$s_{(3,2)\backslash(1,0)} = s_{(2,2)} + s_{(3,1)}$$

$$h_{(2,2)} = s_{(4)} + s_{(3,1)} + s_{(2,2)}$$

$$\left(s_{(3,2)\backslash(1,0)}, h_{(2,2)}\right) = 2$$

Remark

The dimension of the space of symmetric functions is the number of partitions, which is smaller than both the number of possible shapes and the number of possible contents. So there must be relations!

Variations

Suppose we want to count arbitrary SSYT with entries $1, \ldots, k$

Variations

Suppose we want to count arbitrary SSYT with entries $1, \ldots, k$

Or SSYT with entries $0, 1, 2, \cdots$, but we weigh each entry i by q^i.

Variations

Suppose we want to count arbitrary SSYT with entries $1, \ldots, k$

Or SSYT with entries $0, 1, 2, \cdots$, but we weigh each entry i by q^i.

In each case, we know that there exists a symmetric function f (the sum $\chi := \sum_c h_{\nu(c)}$ or a linear combination) such that the answer is $(s_{\lambda \setminus \mu}, f)$. The question is only how to find f.

Variations

Suppose we want to count arbitrary SSYT with entries $1, \dots, k$

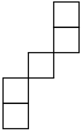
Or SSYT with entries $0, 1, 2, \cdots$, but we weigh each entry i by q^i.

In each case, we know that there exists a symmetric function f (the sum $\chi := \sum_c h_{\nu(c)}$ or a linear combination) such that the answer is $(s_{\lambda\backslash\mu}, f)$. The question is only how to find f.

Approach: count SSYT of some easy shape, recover f and use it for all shapes.

Variations

In the first situation we consider shapes of the form



. The answer is given by the binomial coefficients, so we obtain

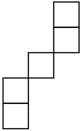
$$(f, e_\lambda) = \prod_i \binom{m}{\lambda_i}.$$

and recover f:

$$f = \sum_\lambda \prod_i \binom{m}{\lambda_i} \omega(m_\lambda)$$

Variations

In the first situation we consider shapes of the form



. The answer is given by the binomial coefficients, so we obtain

$$(f, e_\lambda) = \prod_i \binom{m}{\lambda_i}.$$

and recover f:

$$f = \sum_\lambda \prod_i \binom{m}{\lambda_i} \omega(m_\lambda)$$

also

$$f = \sum_\lambda \prod_i \binom{m + \lambda_i - 1}{\lambda_i} m_\lambda$$

P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A unit interval order is a linearly ordered set $(P, >)$ together with a second order relation $\succ$ called "much greater than" satisfying

$$a \succ b \;\Rightarrow\; a > b, \quad a \succ b \wedge b > c \;\vee\; a > b \wedge b \succ c \;\Rightarrow a \succ c.$$

P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A unit interval order is a linearly ordered set $(P, >)$ together with a second order relation $\succ$ called "much greater than" satisfying

$$a \succ b \;\Rightarrow\; a > b, \quad a \succ b \wedge b > c \;\vee\; a > b \wedge b \succ c \;\Rightarrow a \succ c.$$

Equivalent objects: Dyck path, Hessenberg function.

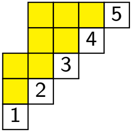
P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A unit interval order is a linearly ordered set $(P, >)$ together with a second order relation $\succ$ called "much greater than" satisfying

$$a \succ b \;\Rightarrow\; a > b, \quad a \succ b \wedge b > c \;\vee\; a > b \wedge b \succ c \;\Rightarrow a \succ c.$$

Equivalent objects: Dyck path, Hessenberg function.



$4 \succ 1, 5 \succ 1$

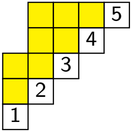
P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A unit interval order is a linearly ordered set $(P, >)$ together with a second order relation $\succ$ called "much greater than" satisfying

$$a \succ b \;\Rightarrow\; a > b, \quad a \succ b \wedge b > c \;\vee\; a > b \wedge b \succ c \;\Rightarrow a \succ c.$$

Equivalent objects: Dyck path, Hessenberg function.



$4 \succ 1, 5 \succ 1$

Slang: if $a < b$, but $a \not\prec b$, i.e. if b is "not too far", we say that a attacks b. Above 1 attacks $2, 3$ and 2 attacks $3, 4, 5$.

P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A P-tableau of some shape $\lambda \setminus \mu$ with entries in some UIO P is a filling of $\lambda \setminus \mu$ by entries of P such that we have

$$\boxed{a}\boxed{b} \Rightarrow a \not\succ b, \qquad \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \Rightarrow a \prec b.$$

P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A P-tableau of some shape $\lambda \setminus \mu$ with entries in some UIO P is a filling of $\lambda \setminus \mu$ by entries of P such that we have

$$\boxed{a \,|\, b} \Rightarrow a \not\succ b, \qquad \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \Rightarrow a \prec b.$$

Some observations:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \text{ is a } P\text{-tableau} \;\Leftrightarrow\; \boxed{b \,|\, a} \text{ is not a } P\text{-tableau}.$$

P-tableaux (after Gessel-Viennot, Shareshian-Wachs,...)

Definition

A P-tableau of some shape $\lambda \setminus \mu$ with entries in some UIO P is a filling of $\lambda \setminus \mu$ by entries of P such that we have

$$\boxed{a \mid b} \Rightarrow a \not\succ b, \qquad \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \Rightarrow a \prec b.$$

Some observations:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \text{ is a } P\text{-tableau} \Leftrightarrow \boxed{b \mid a} \text{ is not a } P\text{-tableau.}$$

$$\begin{array}{|c|c}\cline{1-1} a & \\ \hline & \multicolumn{1}{|c|}{b} \\ \cline{2-2}\end{array} \text{ is in a } P\text{-tableau} \Rightarrow a < b.$$

Definition

A P-tableau of some shape $\lambda \setminus \mu$ with entries in some UIO P is a filling of $\lambda \setminus \mu$ by entries of P such that we have

$$\boxed{a \mid b} \Rightarrow a \nsucc b, \qquad \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \Rightarrow a \prec b.$$

Some observations:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \text{ is a } P\text{-tableau} \Leftrightarrow \boxed{b \mid a} \text{ is not a } P\text{-tableau.}$$

$$\begin{array}{|c|c}\hline a & \multicolumn{1}{|c}{} \\ \hline \multicolumn{1}{c|}{} & b \\ \cline{2-2}\end{array} \text{ is in a } P\text{-tableau} \Rightarrow a < b.$$

Exercise: any two elements in neighboring diagonals know their relative position.

Counting P-tableaux

Definition

The q-weight of a P-tableau is

$$q^{\text{number of pairs } a, b, \text{ such that } a \text{ attacks } b \text{ and } a \text{ is to the left of } b}.$$

Counting P-tableaux

Definition

The q-weight of a P-tableau is

$$q^{\text{number of pairs } a, b, \text{ such that } a \text{ attacks } b \text{ and } a \text{ is to the left of } b}.$$

Theorem (Shareshian-Wachs)

For a finite UIO P there exists a symmetric function χ_P such that for any skew shape $\lambda \setminus \mu$ the q-weighted number of P-tableaux of shape $\lambda \setminus \mu$ is given by

$$(\chi_P, s_{\lambda\setminus\mu}).$$

Counting P-tableaux

Definition

The q-weight of a P-tableau is

$$q^{\text{number of pairs } a, b, \text{ such that } a \text{ attacks } b \text{ and } a \text{ is to the left of } b}.$$

Theorem (Shareshian-Wachs)

For a finite UIO P there exists a symmetric function χ_P such that for any skew shape $\lambda \setminus \mu$ the q-weighted number of P-tableaux of shape $\lambda \setminus \mu$ is given by

$$(\chi_P, s_{\lambda\setminus\mu}).$$

This χ_P is called chromatic symmetric function, or LLT function.

Some examples

UIO: $\mathbb{Z}_{\geq 0}$ with $i \succ j$ if $i > j + 1$.

A sequence $(a_1, a_2, \dots, a_n)$ is a P-tableau of shape (n) iff $a_2 \geq a_1 - 1$, $a_3 \geq a_2 - 1$, and so on. Equivalently,

$$a_{n-1} \leq a_n + 1, \qquad a_{n-2} \leq a_{n-1} + 1, \dots.$$

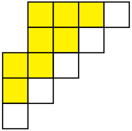
Some examples

UIO: $\mathbb{Z}_{\geq 0}$ with $i \succ j$ if $i > j+1$.

A sequence $(a_1, a_2, \ldots, a_n)$ is a P-tableau of shape (n) iff $a_2 \geq a_1 - 1$, $a_3 \geq a_2 - 1$, and so on. Equivalently,

$$a_{n-1} \leq a_n + 1, \qquad a_{n-2} \leq a_{n-1} + 1, \ldots.$$

If additionally $a_n = 0$, then this is the area sequence of a Dyck path!



$\rightsquigarrow \quad 0, 1, 2, 2, 3 \quad \rightsquigarrow$ | 3 | 2 | 2 | 1 | 0 |.

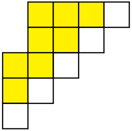
Some examples

UIO: $\mathbb{Z}_{\geq 0}$ with $i \succ j$ if $i > j + 1$.

A sequence $(a_1, a_2, \ldots, a_n)$ is a P-tableau of shape (n) iff $a_2 \geq a_1 - 1$, $a_3 \geq a_2 - 1$, and so on. Equivalently,

$$a_{n-1} \leq a_n + 1, \qquad a_{n-2} \leq a_{n-1} + 1, \ldots.$$

If additionally $a_n = 0$, then this is the area sequence of a Dyck path!



$\rightsquigarrow \quad 0, 1, 2, 2, 3 \quad \rightsquigarrow$ | 3 | 2 | 2 | 1 | 0 |.

Parking functions: similarly. Other shapes: nested Dyck paths.

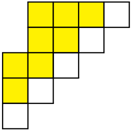
Some examples

UIO: $\mathbb{Z}_{\geq 0}$ with $i \succ j$ if $i > j+1$.

A sequence $(a_1, a_2, \ldots, a_n)$ is a P-tableau of shape (n) iff $a_2 \geq a_1 - 1$, $a_3 \geq a_2 - 1$, and so on. Equivalently,

$$a_{n-1} \leq a_n + 1, \qquad a_{n-2} \leq a_{n-1} + 1, \ldots.$$

If additionally $a_n = 0$, then this is the area sequence of a Dyck path!



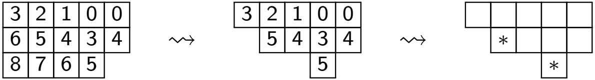
$$\rightsquigarrow \quad 0, 1, 2, 2, 3 \quad \rightsquigarrow \boxed{3}\boxed{2}\boxed{2}\boxed{1}\boxed{0}.$$

Parking functions: similarly. Other shapes: nested Dyck paths.

We could use it to enumerate Dyck paths if only we could single out the tableaux satisfying $a_n = 0$.

A generalization

It is natural! In the context of Young tableaux, the set of cells containing $< k$ is again a skew shape. For P-tableaux it is not so. Take ≤ 5 in



Each column still is a column, but the height can increase, but only if a column ends with a 5.

A generalization

It is natural! In the context of Young tableaux, the set of cells containing $< k$ is again a skew shape. For P-tableaux it is not so. Take ≤ 5 in

3	2	1	0	0
6	5	4	3	4
8	7	6	5	

$\rightsquigarrow$

3	2	1	0	0
	5	4	3	4
		5		

$\rightsquigarrow$

	$*$			
			$*$	

Each column still is a column, but the height can increase, but only if a column ends with a 5.

Definition

A generalized shape is a shape of the form $(\lambda_1, \lambda_2, \dots, \lambda_n)^t$, some of the cells of the shape are marked with stars, so that

1. A star can only be placed in the bottommost cell of a column.
2. We have $\lambda_i \leq \lambda_{i-1}$, but if the i-th column has a star, then $\lambda_i = \lambda_{i-1} + 1$ is also allowed.

Consider a UIO together with a number k so that the k greatest elements are maximal for $\succ$. These will be called *special*. Consider a generalized shape with k stars. Fix a bijection between the stars and the special elements. With these entries fixed, we q-count the P-tableaux.

Consider a UIO together with a number k so that the k greatest elements are maximal for $\succ$. These will be called *special*. Consider a generalized shape with k stars. Fix a bijection between the stars and the special elements. With these entries fixed, we q-count the P-tableaux. Now the condition $a_1 = 0$ can be expressed as a "star-condition".

Consider a UIO together with a number k so that the k greatest elements are maximal for $\succ$. These will be called *special*. Consider a generalized shape with k stars. Fix a bijection between the stars and the special elements. With these entries fixed, we q-count the P-tableaux. Now the condition $a_1 = 0$ can be expressed as a "star-condition".

A Dyck path with a prescribed position of touch-points is the same thing as a tableau of shape

$*$						$*$			

UIO: $\mathbb{Z}_{\leq}$, maximal elements: 0. Dyck path $\rightsquigarrow$ tableau with entries $-a_0, -a_1, \ldots$.

The space V_k is the space of functions symmetric in all but k variables

$$V_k = \mathbb{C}[x_1, x_2, \dots]^{S_\infty}[y_1, \dots, y_k].$$

Endow V_k with a pairing: monomials in y times Schur functions in x form an orthnormal basis.

Theorem

For each UIO P with k special elements there exists an element $\chi_{P,k} \in V_k$, and for each generalized shape λ with k stars there exists an element $\chi_\lambda \in V_k$ so that the count above is given by

$$(\chi_{P,k}, \chi_\lambda).$$

The space V_k is the space of functions symmetric in all but k variables

$$V_k = \mathbb{C}[x_1, x_2, \ldots]^{S_\infty}[y_1, \ldots, y_k].$$

Endow V_k with a pairing: monomials in y times Schur functions in x form an orthnormal basis.

Theorem

For each UIO P with k special elements there exists an element $\chi_{P,k} \in V_k$, and for each generalized shape λ with k stars there exists an element $\chi_\lambda \in V_k$ so that the count above is given by

$$(\chi_{P,k}, \chi_\lambda).$$

$\chi_{P,k}$ is basically the "character of a partial Dyck path". In this context it is defined by counting fillings of collections of vertical bars (= chains in the UIO).

From V_k back to (another) Sym

$$\mathrm{Sym} \xrightarrow{\text{generalization}} V_k \xrightarrow{\text{projection}} \mathbb{C}[y_1, \ldots, y_k] \xrightarrow{\text{symmetrization}}$$

Corresponds to

$$\text{shapes} \xrightarrow{\text{generalization}} \text{generalized shapes} \xrightarrow{\text{specialization}} \text{special shapes}$$

$$\text{contents} \xrightarrow{\text{generalization}} \text{UIOs} \xrightarrow{\text{specialization}} \text{connected UIOs}$$

Definition

A generalized shape is *special* if every row begins with a star. A UIO is *connected* if the attack graph is connected.

From V_k back to (another) Sym

$$\mathrm{Sym} \xrightarrow{\text{generalization}} V_k \xrightarrow{\text{projection}} \mathbb{C}[y_1, \ldots, y_k] \xrightarrow{\text{symmetrization}}$$

Corresponds to

$$\text{shapes} \xrightarrow{\text{generalization}} \text{generalized shapes} \xrightarrow{\text{specialization}} \text{special shapes}$$

$$\text{contents} \xrightarrow{\text{generalization}} \text{UIOs} \xrightarrow{\text{specialization}} \text{connected UIOs}$$

Definition

A generalized shape is *special* if every row begins with a star. A UIO is *connected* if the attack graph is connected.

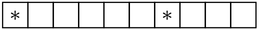
We have two symmetric functions: Shareshian-Wachs's χ_P, and a new function $\chi'_{P,k}$.

Special cases

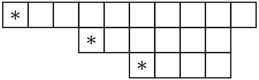
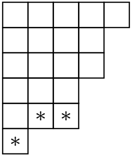
$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline * & & & & & & * & & & \\ \hline\end{array} \rightsquigarrow C_{6,4} = C_6 C_4(1)$$

certain modified Hall-Littlewood polynomial.

Special cases

 $\rightsquigarrow C_{6,4} = C_6 C_4(1)$

certain modified Hall-Littlewood polynomial.

 $\rightsquigarrow$ $\pm$Schur of 

The row lengths turn into the hook lengths of the main diagonal.

Conclusion

Theorem

If some enumeration of Dyck paths/parking functions/UIOs is given by scalar products (f, C_α) for some $f \in \mathrm{Sym}$, then the corresponding enumeration of nested Dyck paths/parking functions/P-tableaux is given by the scalar products $(f, \pm s_\lambda)$.

$$\nabla \widetilde{H}_\lambda[X;q,t] = q^{\sum a_i} t^{\sum \ell_i} \widetilde{H}_\lambda[X;q,t]$$

Expanding ∇f leads to sums of rational functions (like in Eugene's talk)

Shuffle conjecture

$$\nabla \widetilde{H}_\lambda[X; q, t] = q^{\sum a_i} t^{\sum \ell_i} \widetilde{H}_\lambda[X; q, t]$$

Expanding ∇f leads to sums of rational functions (like in Eugene's talk)

But we know that the denominators always cancel out, and sometimes ∇f is even positive!

Shuffle conjecture

$$\nabla \widetilde{H}_\lambda[X;q,t] = q^{\sum a_i} t^{\sum \ell_i} \widetilde{H}_\lambda[X;q,t]$$

Expanding ∇f leads to sums of rational functions (like in Eugene's talk)

But we know that the denominators always cancel out, and sometimes ∇f is even positive!

compositional shuffle conjecture [HHLRU], shuffle theorem [CM]

$$\nabla e_n = \text{enumeration over parking functions}$$

Parking function = Dyck path+labeling of vertical steps

$$e_n = \sum_\alpha C_\alpha, \qquad \nabla C_\alpha = \text{paths have prescribed touch points}$$

Loehr-Warrington conjecture, theorem [BHMPS]

$$\nabla s_\lambda = \text{enumeration over nested parking functions}$$

Loehr-Warrington conjecture, theorem [BHMPS]

$$\nabla s_\lambda = \text{enumeration over nested parking functions}$$

Move nabla to the right

$$(\nabla f, h_\mu) = (f, \nabla^* h_\mu).$$

Loehr-Warrington conjecture

Loehr-Warrington conjecture, theorem [BHMPS]

$$\nabla s_\lambda = \text{enumeration over nested parking functions}$$

Move nabla to the right

$$(\nabla f, h_\mu) = (f, \nabla^* h_\mu).$$

So both conjectures are equivalent to the statement

$$\nabla^* h_\mu = \text{weighted sum of } \chi'_P \text{ over UIOs P.}$$

Loehr-Warrington conjecture

Loehr-Warrington conjecture, theorem [BHMPS]

$$\nabla s_\lambda = \text{enumeration over nested parking functions}$$

Move nabla to the right

$$(\nabla f, h_\mu) = (f, \nabla^* h_\mu).$$

So both conjectures are equivalent to the statement

$$\nabla^* h_\mu = \text{weighted sum of } \chi'_P \text{ over UIOs P.}$$

In fact we have UIOs the submultisets of $\mathbb{Z}_{\leq 0}$, $a \succ b$ if $a \geq b + n$. Each $-a$ is weighted by $t^{\lfloor \frac{a}{n} \rfloor}$.

Loehr-Warrington conjecture, theorem [BHMPS]

$$\nabla s_\lambda = \text{enumeration over nested parking functions}$$

Move nabla to the right

$$(\nabla f, h_\mu) = (f, \nabla^* h_\mu).$$

So both conjectures are equivalent to the statement

$$\nabla^* h_\mu = \text{weighted sum of } \chi'_P \text{ over UIOs P.}$$

In fact we have UIOs the submultisets of $\mathbb{Z}_{\leq 0}$, $a \succ b$ if $a \geq b + n$. Each $-a$ is weighted by $t^{\lfloor \frac{a}{n} \rfloor}$. Also works for ∇^k, and seems to work in the rational case.

Open problems I

About χ_P: Stanley-Stembridge conjecture: χ_P is h-positive.

Open problems I

About χ_P: Stanley-Stembridge conjecture: χ_P is h-positive.

Jeu-de-Taquin on P-tableaux would prove it.

Open problems I

About χ_P: Stanley-Stembridge conjecture: χ_P is *h*-positive.

Jeu-de-Taquin on P-tableaux would prove it.

Shape — content duality: Zelevinsky's pictures would prove it.

Open problems I

About χ_P: Stanley-Stembridge conjecture: χ_P is h-positive.

Jeu-de-Taquin on P-tableaux would prove it.

Shape — content duality: Zelevinsky's pictures would prove it.

Are there some algebraic structures on P-tableaux, similar to the affine Hecke algebra representations with basis the usual skew tableaux?

Open problems II

About χ'_P: It seems there is positivity involving canonical basis in the affine Hecke algebra. Polynomials in $y_1, \ldots, y_k$ correspond to

$$\mathrm{AffHecke}_k \cdot \sum_{\pi \in S_n} T_\pi.$$

Open problems II

About χ'_P: It seems there is positivity involving canonical basis in the affine Hecke algebra. Polynomials in $y_1, \ldots, y_k$ correspond to

$$\mathrm{AffHecke}_k \cdot \sum_{\pi \in S_n} T_\pi .$$

Canonical basis in V_k?

Open problems II

About χ'_P: It seems there is positivity involving canonical basis in the affine Hecke algebra. Polynomials in $y_1, \ldots, y_k$ correspond to

$$\mathrm{AffHecke}_k \cdot \sum_{\pi \in S_n} T_\pi.$$

Canonical basis in V_k?

Why strange signed Schur positivity?

Geometry

For χ_P there is a geometric interpretation by Shareshian-Wachs. Hessenberg varieties (Julianna's talk).

Geometry

For χ_P there is a geometric interpretation by Shareshian-Wachs. Hessenberg varieties (Julianna's talk).

Given P, k and a regular semisimple γ consider space of flags satisfying

1. $\gamma F_i \subset F_{h(i)}$
2. F_{n-k} does not contain any coordinate line.

For χ_P there is a geometric interpretation by Shareshian-Wachs. Hessenberg varieties (Julianna's talk).

Given P, k and a regular semisimple γ consider space of flags satisfying

1. $\gamma F_i \subset F_{h(i)}$
2. F_{n-k} does not contain any coordinate line.

Example: $\mathbb{P}^{n-1}$ without coordinate hyperplanes. Obtain $(\mathbb{C}^*)^{n-1}$, the cohomology is the exterior algebra.

Geometry

For χ_P there is a geometric interpretation by Shareshian-Wachs. Hessenberg varieties (Julianna's talk).

Given P, k and a regular semisimple γ consider space of flags satisfying

1. $\gamma F_i \subset F_{h(i)}$
2. F_{n-k} does not contain any coordinate line.

Example: $\mathbb{P}^{n-1}$ without coordinate hyperplanes. Obtain $(\mathbb{C}^*)^{n-1}$, the cohomology is the exterior algebra.

As S_n-rep we have H^i is the i-th hook.

Conjecture

Some spaces have the property: for a partition λ the corresponding S_n-rep appears only in $H^{\iota(\lambda)}$ where $\iota(\lambda)$ is the number of cells under the main diagonal.