Catalan Combinatorics

OPAC 2022

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10 YEARS AGO

American Institute of Mathematics

Rational Catalan combinatorics

December 17 to December 21, 2012

at the

American Institute of Mathematics, Palo Alto, California

organized by

Drew Armstrong, Stephen Griffeth, Victor Reiner, and Monica Vazirani

This workshop, sponsored by AIM and the NSF, will be devoted to understanding the interaction between new developments in algebra and combinatorics. In particular, it will focus on combinatorial objects counted by generalizations of Catalan numbers and their interaction with the representation theory of Cherednik algebras.
This Year

POSITROIDS, KNOTS, AND q,t-CATALAN NUMBERS

PAVEL GALASHIN AND THOMAS LAM

ABSTRACT. We relate the mixed Hodge structure on the cohomology of open positroid varieties (in particular, their Betti numbers over \( \mathbb{C} \) and point counts over \( \mathbb{F}_q \)) to Khovanov-Rozansky homology of associated links. We deduce that the mixed Hodge polynomials of top-dimensional open positroid varieties are given by rational \( q, t \)-Catalan numbers. Via the curious Lefschetz property of cluster varieties, this implies the \( q, t \)-symmetry and unimodality properties of rational \( q, t \)-Catalan numbers. We show that the \( q, t \)-symmetry phenomenon is a manifestation of Koszul duality for category \( \mathcal{O} \), and discuss relations with open Richardson varieties and extension groups of Verma modules.

Nathan Williams
Parking analogue of Galashin-Lam?
To: Pavel Galashin

Hi Pavel,

n=3
R.<q> = QQ[]
W = WeylGroup(['A',n,1])
KL = KazhdanLusztigPolynomial(W, q)
f=KL.R(1,W.from_reduced_word(list(range(n+1))*n))/(q-1)^n*(2^n)
f==sum((q^i for i in range(n+1)))*n-1)

Best,
Nathan
February 4, 2022: Galashin-Lam’s R-polynomial via Hecke algebra, Opdam’s trace formula, and Haglund’s Tesler matrix identity.

NOT TODAY.

CONTINUED EXPERIMENTS into March, which led to THIS TALK.

We had been looking for these results since the AIM conference 10 YEARS AGO.
0. Catalan Numbers
DEF The **Catalan numbers** are the integers

\[
\text{Cat}(n) = \frac{1}{2n+1} \binom{2n+1}{n}
\]

EX 1, 1, 2, 5, 14, 42, 132, ...

\(\text{Cat}(n)\) counts: noncrossing partitions, triangulations, Dyck paths, etc, etc, etc, etc, etc, ...

REF Pak, "History of Catalan Numbers"  
Stanley, "Catalan Numbers"
"THM" (Folklore)

Just about every combinatorial object is Catalan.
**DEF** The **Catalan numbers** are the integers

\[ \text{Cat}(n) = \frac{1}{2n+1} \binom{2n+1}{n} = \frac{1}{n+1+n} \binom{n+1+n}{n} = \prod_{i=1}^{n-1} \frac{(n+1)+i}{i+1} \]

**PARAMETER**

\[ \approx \]

**TYPE**

Ref: Stanley, "Catalan Numbers"
I. Reflection Groups

Diagram:
- Symmetric Group
- Cat(n)
- Reflection groups
- Cartan
- Coxeter groups
- Reflection
- r
- p
- Parameter

Additional notes:
- каталган
- Coxeter
- Cartan
- reflection
**Type A**

**Philosophy:** "\( \mathcal{G}_n \) is \( SL_n(\mathbb{F}_q) \) at \( q = 1 \)"

\[ |SL_n(\mathbb{F}_q)| = (q-1)^n \cdot \prod_{i=1}^{n} (q^i - 1) \]

- Lie group
- Braid group
- Hecke algebra
- Affine symmetric group

- \( SL_n(\mathbb{F}_q) \) Tymoczko's talk
- \( B_n \) Gorsky's talk
- \( H_n \) Mellit's talk
- \( \widetilde{G}_n \) Speyer's talk
WEYL GROUPS ($\mathbb{Q}$)

- connected reductive group over $\overline{\mathbb{F}_p}$, Frobenius $F$
- Weyl group
- Braid group
- Hecke algebra
- Affine Weyl group

$G
\quad W = N_G(T)/T$

**Philosophy:** "$W = G^F$ at $q = 1$" (Tit}s)

$B_w = \pi_1(V^m_g/W)$

$H_w = $ quotient of $\mathbb{C}[B_w]$

$\tilde{W} = W \times \mathbb{Q}'$
Theorem: The list of irreducible Weyl groups is:

\[ A_n \quad E_6 \quad F_4 \]
\[ B_n \quad E_7 \quad G_2 \]
\[ C_n \quad E_8 \]
\[ D_n \]

Ref: Coxeter, "The complete enumeration of finite groups of the form \( r^2 = (i_1 r)^2 = 1 \)" 1935
**Coxeter Groups (R)**

**DEF** A Coxeter system \((W, S)\) is a group \(W\) with presentation \(W = \langle s_1, s_2, \ldots, s_n | (s_i s_j)^{m_{ij}} = id \rangle\).  

- \(S\) is the set of simple reflections.  
  (Coxeter groups act as reflection groups on \(R^2\) with corresponding hyperplane arrangement \(H_w\))

- Braid group \(B_w = \pi_1 (V_{reg} / W)\)

- Hecke algebra \(H_w = \text{quotient of } \mathbb{C}[B_w]\)

- Affine Weyl group \(\tilde{W}\)

- Lie group

**Ref** Hiller, “Geometry of Coxeter groups”
Humphreys, “Reflection groups and Coxeter groups”
Björner & Brenti, “Combinatorics of Coxeter Groups”
**CLASSIFICATION: COXETER GROUPS**

**THM (Coxeter)**

The list of finite irreducible Coxeter groups is:

- $A_n$...
- $E_6$
- $F_4$
- $B_n$
- $E_7$
- $H_3$
- $D_n$
- $E_8$
- $H_4$
- $I_2(m)$

Ref: Coxeter, "The complete enumeration of finite groups of the form $r^2 = (i_j)^{n_j} = 1"$ 1935
**Complex Reflection Groups (C)**

**DEF** A complex reflection group is a group $W \subseteq \text{GL}_n(C)$ generated by complex reflections.

- Braid group
- Simple reflections
- Hecke algebra
- Affine Weyl group
- Lie group (spetses)

**REF**
- Shepherd, Todd. “Finite Unlary Reflection Groups.” 1953
The list of finite irreducible complex reflection groups is:

- $G(m,p,n)$
- $G_4$
- $G_5$
- $G_37 = E_8$

**THM** (Shephard, Todd)

**Ex**

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**REF**
- Broué, Malle, Rouquier, "On Complex Reflection Groups and Their Associated Braid Groups." 1994
- Broué, Malle, Rouquier, "Complex reflection groups, braid groups, Hecke algebras." 1998
- Shephard, Todd. "Finite Unitary Reflection Groups." 1953
IN Variant Theory AND NUMEROLOGY

$W$ acts on $\mathbb{C}^n = \text{span}_\mathbb{C} \{x_1, \ldots, x_n\}$, hence on $\mathbb{C}[x_1, \ldots, x_n]$.

THM (Chevalley) Let $W \leq \text{GL}_n(\mathbb{C})$. Then

$W$ is a complex reflection group iff $\mathbb{C}[x_1, \ldots, x_n]^W = \mathbb{C}[f_1, \ldots, f_n]$.

DEF Let $\deg f_i = d_i$ with $d_1 \leq d_2 \leq \ldots \leq d_n$. ($h = d_n$ is the Coxeter number, $e_i = d_i - 1$ are the exponents)

EX $\mathbb{C}_n$ in $\mathbb{C}^{n-1}$ has invariant polynomials, power sum, elementary, homogeneous, but always $\deg f_i = i+1$. Schur, monomial, forgotten, …

REF Chevalley, Invariants of finite groups generated by reflections.
THE GOLD STANDARD

\[ \mathbb{Z}/\mathbb{R}/\mathbb{C} \text{- UNIFORM} \] definitions & proofs for reflection groups.

“does not appeal to the \( \mathbb{Z}/\mathbb{R}/\mathbb{C} \text{-classification} \)”

\[
\begin{align*}
\text{Ex} & \quad |w| = \prod_{i=1}^{n} d_i \\
(\text{i}) & \quad \text{Hilb}(C[x_1, \ldots, x_n]^W) = \prod_{i=1}^{n} \frac{1}{1-td_i} = \frac{1}{|w|} \sum_{\omega \in \mathbb{W}} \frac{1}{\det(1-tw)} \\
(\text{ii}) & \quad \text{multiply by } (1-t)^n: \quad \prod_{i=1}^{n} \frac{1}{[d_i]} = \frac{1}{|w|} (1 + (1-t)^*x) \\
(\text{iii}) & \quad \text{set } t \rightarrow 1
\end{align*}
\]
II. $\text{Cat}(\mathbb{W})$

- type
- complex reflection groups
- Coxeter groups
- Weyl groups

Symmetric group

- $\mathbb{C}$
- $\mathbb{R}$
- $\mathbb{Q}$

- parameter
- $p$ rational
- $mh \pm 1$
- Fuss-Catalan

$h+1$ Catalan
DEF The Coxeter-Catalan numbers are the integers

\[ \text{Cat}(W) = \prod_{h=1}^{n} \frac{h+1 + e_i}{d_i}. \]

EX \[ \text{Cat}(n) = \text{Cat}(G_\nu) = \prod_{i=1}^{n-1} \frac{(n+1) + i}{i+1}. \]

RECALL "THM" (Folklore) Just about every combinatorial object is Catalan.
THM (Reading, Shi/Callini-Papi)

Only TWO Coxeter-Catalan families:

NC
NONCROSSING

- noncrossing partitions
- clusters
- sortable elements

Cambrian recurrence
Coxeter/well-generated
Dependent on Coxeter element
Hard to change parameter

NN
NONNESTING

- nonnesting partitions
- dominant Shi regions
- coroots in \((\mathfrak{h}+1)\wedge\mathfrak{a}\)

Uniform enumeration
Weyl
Easy to change parameter
III. NONCROSSING PARTITIONS

- type
- complex reflection groups
- Coxeter groups
- Weyl groups
- noncrossing partitions
- $\mathbb{C}$
- $\mathbb{IR}$
- $\mathbb{Q}$
- $\text{Cat}(n)$
- $p$ rational
- $mh \pm 1$
- Fuss-Catalan
- Catalan
IR-TYPE HISTORY OF NONCROSSING PARTITIONS


1993 - Montenegro. The fixed point non-crossing partition lattices

1995 - Reiner. Non-crossing partitions for classical reflection groups

1997 - Birman, Ko, Lee. A new approach to the word problem in the braid groups

2002 - Brady, Watt. K(π,1)'s for Artin groups of finite type

2002 - Picantin. Explicit presentations for the dual braid monoids

2003 - Bessis. The dual braid monoid
**R - Noncrossing Partitions**

**Def** \( NC(n) = \) noncrossing (set) partitions ordered by refinement.

\[
NC(3) = \quad \begin{array}{c}
\begin{array}{c}
\text{(123)} \\
\text{3} \\
\text{2} \\
\text{1}
\end{array}
\end{array}
\quad \subseteq \mathcal{G}_3
\]
**DEF** \( \text{NC}(n) = \) noncrossing (set) partitions ordered by refinement.

\[
\text{NC}(4) = (12(34)) \leq G_4
\]
**R-NONCROSSING-PARTITIONS**

**DEF** The reflections of $W$ are all conjugates of simple reflections:

$$T = \{ wsw' \mid s \in S, w \in W \}.$$  

**DEF** A Coxeter element $c$ is a product of all simple reflections in some order.

**EX** In $S_n$, $T = \{ C(i, j) \mid 1 \leq i < j \leq n \}$.  

$c = (12 \ldots n)$ \text{ the long cycle}$\$

**FACT** The eigenvalues of $c$ in the reflection representation are $\xi \xi_2 \xi_3 \ldots \xi_n$.

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**REF** Bessis. The dual braided monoid  
Reiner. Noncrossing partitions for classical reflection groups.
**Def** The noncrossing partition lattice is the interval \( NC_e(w) = [e, c]_T \) in the oriented Cayley graph of \((w, T)\). Called absolute order, denoted \( \leq_T \).

**Ex** \( NC_{(12\ldots n)}(S_n) \equiv NC(n) \) via cycles.

Ref: Bessis, The dual braid monoid. Rainer, Noncrossing partitions for classical reflection groups.
SUBWORDS & CLUSTERS

**THM** The subwords of $c_{W_0(c)}$ that start at $e$ with $n$ stays and end at $w_0$ are in bijection with $NC_{e}(W)$.  

**EX** $G_3$

![Diagram showing subwords and cluster exchange graph]

**REF:**
- Knutson, Miller. Subword complexes in Coxeter groups.
- Pilaud, Stump. Brick polytopes of spherical subword complexes and generalized associahedron.
**SUBWORDS & CLUSTERS**

**Ex** $G_3$

**BISECTION**: replace stays with colored inversions (root config)

$st(12)(23)$

$sts(12)(23)$

$st(23)(13)s$

$s(13)(12)ts$

$(12)tst(23)$

$c_{w_0}(c) = ssstst$

REF: Knutson, Miller. Subword complexes in Coxeter groups.
Caballero, Labbe, Stump. Subword complexes, cluster complexes, and generalized multi-associahedra.
Pilaud, Stump. Brick polytopes of spherical subword complexes and generalized associahedra.
**IR-NONCROSSING-PARTITIONS**

\[ |NC_c(W)| = \text{Cat}(W) = \prod_{i=1}^{n} \frac{d_i + 1 + e_i}{d_i} \]

**Proof is NOT UNIFORM:** combinatorial models + computer checks

- (classical types)
- (exceptional types)

Only TWO Coxeter-Catalan objects

In particular, the number of clusters in a cluster algebra of finite type was NOT UNIFORMLY proven to be counted by \( \text{Cat}(W) \).
However...

J. Michel recently found a uniform proof for Weyl groups for a related problem (factorizations of c into reflections).

(i) Chapuy- Schimpf formula

(ii) Frobenius character-theoretic method in Hecke algebra

(iii) Deligne-Lusztig theory.

REF: Chapuy & Schimpf. Counting factorizations of Coxeter elements into products of reflections.

Michel: “Case-free” derivations for Weyl groups of the number of reflection factorizations of a Coxeter element.
Problem 1: uniformly prove \( |NC_{\mathfrak{w}}| = \prod_{i=1}^{n} \frac{h+d_i}{d_i} \).

Diagram:
- Type
- Non-crossing partitions
- Complex reflection groups
- Coxeter groups
- Weyl groups
- \( \mathbb{C} \)
- \( \mathbb{R} \)
- \( \mathbb{Q} \)
- \( \text{Symmetric group} \)
- \( \text{Cat}(n) \)
- Parameter
- \( p \) rational
- \( mh \pm 1 \) Fuss-Catalan
- \( h+1 \) Catalan
IV. WHY THE FUSS?

- Fuss-noncrossing partitions
- Complex reflection groups
- Coxeter groups
- Weyl groups
- Type
- $\text{Cat}(n)$: Symmetric group
- Parameter $p$: rational
- $\text{Fuss-Catalan}$
Exploring the History of Noncrossing Partitions

1971 - Kremeras. Sur les partitions non croisées d'un cycle.


Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups

Drew Armstrong

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**DEF**  \[ NC_e^m(W) = \sum_{m\text{-multichains in } NC_e(CW)} \]  

**THM (Armstrong)**  \[ |NC_e^m(W)| = \prod_{i=1}^n \frac{m^d_i + 1}{d_i} \quad \text{(Coxeter-Fuss-Catalan number)} \]

**EX**  \[ NC_e^m(G_n) \] are the \( m \)-divisible noncrossing partitions.

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**REF**  
Eulner, Chain Enumerators and noncrossing partitions  
Armstrong, Generalized noncrossing partitions & combinatorics of Coxeter groups
WHY THE FUSS?

The subwords of $c w_n^m(c)$ that start at $e$ with $n$ stays and end at $w_0^n = (12)_n s t$ are in bijection with $NC_e^m(W)$. 

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Cataland: Why the Fuss?

Christian Stump
Hugh Thomas
Nathan Williams

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Can define $NC_c^m(W)$, counted by \( \prod_{i=1}^{n} \frac{mh-1+e_i}{d_i} \).

And then you get stuck. For 10 years.

BUT...
BUT... \( \prod_{i=1}^{n} \frac{p+e_i}{d_i} \) is ALWAYS an integer for \( \gcd(p, h) = 1 \).

**Problem (D. Armstrong, n 2012):**

What NC object is counted by \( \prod_{i=1}^{n} \frac{p+e_i}{d_i} \)?

- fractional multichains?
- support conditions?
- subwords?
PROBLEM 2: find rational noncrossing partitions

\[ \prod_{i=1}^{n} \frac{p + e_i}{d_i} \]

- type
- complex reflection groups
- Coxeter groups
- Weyl groups
- Symmetric group
- \( \text{Cat}(n) \)
- \( mh \pm 1 \)
- Fuss-Catalan
- \( h+1 \)
- parameter
- \( p \) rational
Only two types of families: noncrossing & nonnesting

V. Nonnesting Partitions

- Type
- Complex reflection groups
- Coxeter groups
- Weyl groups

Cat(n) - Symmetric group

Parameter
- $p$ rational
- $mh \pm 1$
- Fuss-Catalan
- Catalan

Examples:
- $h+1$
- $h$
Nonnesting partitions in $G_n$:

$\text{Ex} \quad \text{NN}(G_n) \equiv \text{nonnesting (set) partitions (Dyck paths)}$

$\text{NN}(S_3) \equiv \text{(diagram of partitions)}$
DEF \( \text{NN}(w) = \sum \text{anti-chains in the positive root poset } \Phi^+ \). (Postnikov)

Weyl group!

THM \( \text{NN}(w) \) is in bijection with coroot pts in \((\chi+1)A_n\). (Cellini-Papi)

\[ S \text{ is a Catalan #!} \]

REP Shi: Sign types corresponding to an affine Weyl group
Reiner: Noncrossing partitions of classical reflection groups
Hughes: Conjectures of the Quotient Ring by Dipendu Hembram
Cellini, Papi: All Hilbert ideals of a Borel subalgebra II
**Nonnesting Partitions**

**DEF** For $\gcd(p, h) = 1$, $NN^{(p)}(w) = \sum_{\text{coroot pts in } PA_0^2}$

**THM (Haiman)** $|NN^{(p)}(w)| = \prod_{i=1}^{n} \frac{p + e_i}{d_i}$. 

**REF** Haiman, Conjectures on the Quotient Ring by Disjoint Harmonics
OPEN PROBLEM 3: Find nonnesting partitions for complex reflection groups.

OPEN PROBLEM 4: Uniform bijection NC \leftrightarrow NN. I have a candidate using toggles: type A proven by Lacim!

\{ Florian Aigner, Benjamin Dequènèé, Gabriel Frieden, Alessandro Iacu, Hugh Thomas \}

\text{parameter}

\text{rational}

\text{Fuss-Catalan}

\text{Cat}(n)

\text{Symmetric group}

\text{type}

\text{complex reflection groups}

\text{C}

\text{IR}

\text{Q}

\text{Weyl groups}

\text{Catalan}

\text{mh+1}

\text{nh+1}

\text{type}

\text{complex reflection groups}

\text{C}

\text{IR}

\text{Q}

\text{Weyl groups}

\text{Catalan}

\text{mh+1}

\text{nh+1}

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\text{IR}

\text{Q}

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\text{Catalan}

\text{mh+1}

\text{nh+1}

\text{Fuss-Catalan}

\text{parameter}

\text{rational}

\text{Cat}(n)

\text{Symmetric group}
RATIONAL NONCROSSING OBJECTS

Galashin, Lam, Trinh, W.
- $\mathbb{R}$-CLOSED: find rational noncrossing partitions $\text{NC}_c^{(p)}(W)$.
- $\mathbb{Q}$-CLOSED: uniformly prove $|\text{NC}_c^{(p)}(W)| = \prod_{i=1}^n \frac{p+q}{i}$. 

Diagram:
- Type
- Complex reflection groups
- Coxeter groups
- Weyl groups
- Symmetric group
- Parameter
- $\mathbb{R}$
- $\mathbb{Q}$
- $h+1$ Catalan
- Fuss-Catalan
- $p$ rational
- $nh \pm 1$
Relative Position in the Flag Variety

Fix $G$ a connected reductive group over $\overline{\mathbb{F}}_p$ with Frobenius $F$.

For $B_1, B_2$ Borel subgroups, write $B_1 \rightarrow B_2$ when

$$(B_1, B_2) \in \Xi(\mathbb{P}B_+, \mathbb{P}B_+) : J \in G \not\exists$$

unique

Diagram:

- $B_+$
- $s \rightarrow B_+$
- $t \rightarrow B_+$
- $s \rightarrow t$
- $t \rightarrow s$
- $s \rightarrow w_0 \rightarrow t$
- $t \rightarrow s$
- $s \rightarrow t$
- $s \rightarrow w_0$
- $t \rightarrow w_0$
CATALAN VARIETIES (WHAT THE HECK?)

Write $c = s_1 s_2 \ldots s_n$ for a Coxeter element.

**DEF** $NCV_c^{(p)}(w) = \{ B_0 \overset{s_1}{\rightarrow} B_1 \overset{s_2}{\rightarrow} \ldots B_n \overset{s_m}{\rightarrow} \ldre B_n \ldresse B_0 \}$

Think "c@$p$"

**THM** (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

(OBA UNIFORM!)

$$|NCV_c^{(p)}(w)| = (q-1)^n \prod_{i=1}^{n} \frac{[p + q_i]}{[d_i]}$$
CATALAN VARIETIES (WHAT THE HECKE?)

THM (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

$$|NCV_c^{(n)}(W)| = (q-1)^n \prod_{i=1}^{n} \frac{[p + e_i]}{[p]}.$$

PROOF METHOD

(i) Hecke algebra
(ii) Character-theoretic method
(iii) Deligne-Lusztig theory

similar to J. Michel’s proof for the Chapuy–Stump formula

+ Gordon, Griffith (i) Connection to rational Cherednik algebra

REF
Gordon, Griffith. Catalan numbers for complex reflection groups.
Trinh. From the Hecke category to the Unipotent locus.
**CASE-BY-CASE PROOF VIA LOW-BROW COMPUTATIONS**

**DEF** The **Hecke algebra** $H_W$ is the complex associative algebra with basis $\bigoplus T_w \mathbb{C}^{w \in W}$ and relations induced by

1. $T_u T_v = T_{uv}$ if $l(u) + l(v) = l(uv)$ and
2. $(T_s + \mathbb{C}) (T_s - 1) = 0$ for $s \in S$.

**DEF** The **trace** $\text{tr}: H_W \to \mathbb{C}[q, q^{-1}]$ is given by

$$\text{tr}(T_w) = \begin{cases} 1 & \text{if } w = e \\ 0 & \text{otherwise} \end{cases}$$
Case-By-Case Proof Via Low-Brow Computations

**Fact 1** $|NCV_c^{(p)}(w)| = q^n \text{tr}(T_c^{-p})$ (Deadhar recurrence)

Hecke algebra trace $\text{tr}(T_w) = \sum_{i=1}^{n} 1$ if $w=1$

**Fact 2**

$$\text{tr}(T_c^{-p}) = \sum_{\chi \in \text{Irr}(W)} \frac{1}{S_{\chi}(q)} \chi_1(T_c^{-p})$$

Schur elements (formulas + tables exist) thank you Gote Pfeiffer!

**Fact 3**

$$\chi_2(T_c^{-p}) = q^{\frac{ph_{\chi}}{h}} - q^n \chi(c) \quad \text{for } \gcd(p, h) = 1$$

Relative Coexter number

**Fact 4**

$$\chi(c) = 0 \text{ on all but only } h \text{ many irreps } \sum_{i=1}^{h}$$

$$|NCV_c^{(p)}(w)| = \sum_{i=1}^{h} \frac{q^{\phi_i/h}}{S_{\chi_i}(q)} \chi_i(c)$$

Ref: Geck, Pfeiffer. Characters of Finite Coxeter groups and Iwahori-Hecke algebras
Now can evaluate \textit{CASE-BY-CASE}:

For $G_n$, this is a specialization of a computation of V. Jones:

$$|NCV_c(q)(G_n)| = \frac{1}{[n]!} \sum_{i=1}^{n} q^{P(n-i)+(n-i+1)} \left[\frac{n-1}{i-1}\right] (-1)^i$$

$$= (q-1)^{n-1} \prod_{i=1}^{n} \frac{[P + z_i]}{[d_i]} \quad (\text{by } q\text{-binomial theorem})$$

\textit{REF} Jones. Hecke algebra representations of braid groups and link polynomials GAP 3 with CHEVIE
CATALAN VARIETIES (WHAT THE HECKE?)

THM (Galashin, Lam, Trinh, W.) Over $\mathbb{F}_q$

$$|NCV_c^{(q)}(w)| = (q-1)^n \prod_{i=1}^{n} \frac{[p+e_i]}{[\lambda_i]}.$$ 

SHOW ME THE COMBINATORICS!
Consider the relative position of $B_-$ and $B_c$ for an element of $NC^c(W)_c$:

$B_0 \overset{s_1}{\rightarrow} B_1 \overset{s_2}{\rightarrow} \ldots \overset{s_n}{\rightarrow} B_n \overset{s_1}{\rightarrow} B_{n+1} \overset{s_2}{\rightarrow} \ldots \overset{B_{n+p}}{\rightarrow} B_{n+p}$

$\uparrow u_0 w_0 \uparrow u_1 w_0 \uparrow u_n w_0 \uparrow \uparrow u_{n+p} w_0$

Then $u_0, u_1, \ldots, u_{n+p}$ encodes:

(i) a subword of $c^p$

(ii) that starts and ends at $c$

(iii) can stay, but must go down when possible.

$D(c^p, c)$

no odd colors on stays
Ex. $W = G_3$, $p = 4$
$c = st$

Elements of $D(c^4, e)$ with 2 stays

Start & end at $e$, no odd colors are distinguished on stays

Diagram illustrating the elements with 2 stays.
Example: $W = G_3$, $p = 4$

$c = st$

Elements of $D(c^4, e)$ with 2 stays
Start & end at $e$, no odd colors

Diagram:

- $ststst(st(12)(23))$
- $ststst(23)(13)t$
- $ststst(13)(12)t$
- $ststst(13)(12)t$
- $ststst(13)(12)t$
- $ststst(13)(12)t$
- $ststst(13)(12)t$
- $ststst(13)(12)t$
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- $ststst(13)(12)t$
- $ststst(13)(12)t$
- $ststst(13)(12)t$
**DEODHAR DECOMPOSITION**

**DEF**
\[ e_u = \# \text{ stays} \]
\[ d_u = \# \text{ descents} \]

**THM (Deodhar)**
\[ |NCV_c^{(p)}(W)| = \sum_{u \in D(c^p, e)} (q-1)^{e_u} q^{d_u} \]

So
\[ \sum_{u \in D(c^p, e)} (q-1)^{e_u} q^{d_u} = (q-1)^n \prod_{i=1}^{n} \frac{[p + q - e_i]}{[d_i]} \]

want those \( u \) for which \( e_u = n \) (minimal \# stays), then send \( q \to 1 \).
From the Deodhar Decomposition to Combinatorics

\[ \text{DEF } \mathcal{NC}_c^{(p)}(w) = \left\{ u \in D(c^p, e) : e_u = n \right\} \]

= distinguished subwords with exactly \( n \) stays

Compute:

\[
\sum_{u \in \mathcal{NC}_c^{(p)}(w)} q^{d_u} + \sum_{u \in D(c^p, e), e_u > n} (q-1)^{e_u-n} q^{d_u} = \prod_{i=1}^{n} \frac{[p + e_i]}{[d_i]}
\]

So at \( q = 1 \):

\[ |\mathcal{NC}_c^{(p)}(w)| = \prod_{i=1}^{n} \frac{p + e_i}{d_i} \]
I should convince you that $NC_c^{(p)}(w)$ is a noncrossing object.

**THM** $NC_c^{(m+1)}(w)$ is in bijection with $NC_c^m(w)$.

**Proof** Halve the colors.
\[ \text{EX } G_3 \quad m = 1 \]

\[ \text{NC}_c^{(m+1)}(w) = \text{Subwords for } w_0 \text{ in } c_0^2 c \text{ with } n \text{ stays} \]

\[ \text{NC}_c^m(w) = \text{Subwords for } w_0 \text{ in } c_0^2 c \text{ with } n \text{ stays} \]
EX \( G_3 \): \( m = 1 \)

\[ NC_c^{(m+1)}(W) \]

Subwords for \( w_0 \) in \( cW_0 c^2 \) with \( n \) stays and no odd colors

\[ NC_c^m(W) \]

Subwords for \( w_0 \) in \( cW_0 c^2 \) with \( n \) stays
What object is counted by $\prod_{i=1}^{n} \frac{p + e_i}{d_i}$? (D. Armstrong, n 2012)

$\text{THM}$

$\text{NC}_c^{(p)}(w)$.

(Galashin, Lam, Trinh, W., 2022)
OPEN PROBLEM 5: Find combinatorial models for $\text{NC}^{(p)}_c(n)$. 

Diagram:
- Type: Complex reflection groups, Coxeter groups, Weyl groups
- Parameter: $p$ rational, $qh \leq 1$ Fuss-Catalan
- Symmetric group $\text{Cat}(n)$
WHAT ABOUT PARKING STRUCTURES™?
\[ \text{YES!} \]

\[ \text{THM (Galashin, Lam, Trinh, W.) Over } \mathbb{F}_2 \]

\[ |\text{NCPV}_c^{(p)}(w)| = (q-1)^n [p]^n. \]

Again, get combinatorial objects as subwords with exactly \( n \) skips.
OPEN PROBLEMS

OPEN PROBLEM 1: Uniformly prove \( |\text{NC}^{(p)}_c(w)| = \prod_{i}^{p+e_i} \).  

OPEN PROBLEM 2: Find rational noncrossing partitions \( \text{NC}^{(p)}_c(w) \).

OPEN PROBLEM 3: Find nonnesting partitions for complex reflection groups.

OPEN PROBLEM 4: Uniform bijection \( \text{NC}^{(p)}_c(w) \leftrightarrow \text{NN}^{(p)}(w) \) (toggle bijection for \( p=n+1 \)).

OPEN PROBLEM 5: Find combinatorial models for \( \text{NC}^{(p)}_c(w) \) in classical types.

OPEN PROBLEM 6: Follow Galashin's "recipe for success" (compute mixed Hodge cohomology).
OPEN PROBLEMS FOUND WHILE MAKING THESE SLIDES

OPEN PROBLEM 7: Show that the restriction of $Camb_e^2(W)$ to the Deodar words is isomorphic to $NC_e(W)$. What happens for $Camb_e^{2m}(W)$?
**BONUS PROBLEMS (R-polynomials)**

**BONUS PROBLEM 1:** In \( \widetilde{G}_n \), show

\[
R_{\lambda, \mu} (q) = (q-1)^{\mu_1 - \mu_n} \sum_{\mu \geq n} f_{\mu, \lambda} q^\text{stat}(\mu)
\]

**BONUS PROBLEM 2:** In \( G_n \), show

\[
R_{\lambda, w_0} (q) = (q-1)^{n} \sum_{\mu \geq n} \sum_{T \in \text{SYT}(\mu)} q^\text{stat}(T)
\]

**BONUS PROBLEM 3:** In \( \tilde{G}_n \), let

\[
\lambda = \sum_{i=1}^{n+1} \alpha_i \xi_i = \lambda_+ - \lambda_-
\]

with \( \alpha_1 > \alpha_2 > \cdots > \alpha_{n+1} \geq 0 \).

Show

\[
R_{\lambda_+, \lambda_-} (q) = (q-1)^{n+1} \prod_{i=1}^{n+1} \left( \frac{q^{\alpha_i} - q^{\alpha_{i+1}}}{q-1} \right)
\]

**BONUS PROBLEM 4:** Fix \( \frac{m}{n} > 1 \). In \( \widetilde{G}_m \), define

\[
w_{m,n} = (s_{m-n+1} \cdots s_{m-1} s_0 s_{m-n} \cdots)
\]

Show

\[
R_{\lambda, w_{m,n}} (q) = (q-1)^{n+m-1} \left[ m \right]_q^{n-1}
\]
THANK YOU!!!