What is a combinatorial interpretation?

(joint work with Christian Ikenmeyer)

Open Problems in Algebraic Combinatorics (Minneapolis, 2022)
Plan of the talk: *four deep questions*

1) Why do we care about combinatorial interpretations?
2) Why are people so *positive* about their existence against both evidence, reason and experience?
3) What *are* combinatorial interpretations?
4) How can we prove that they don’t exist?

What is in #P and what is not?

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Most wanted combinatorial interpretations

- **Kronecker coefficients** $g(\lambda, \mu, \nu) \in \mathbb{N}$

  $$\chi^\mu \cdot \chi^\nu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu) \chi^\lambda \quad \text{where} \quad \mu, \nu \vdash n$$

  describe **tensor products** of irreducible $S_n$-reps
generalize **Littlewood–Richardson coefficients**

- **plethysm coefficients** $a_\lambda(\mu, \nu) \in \mathbb{N}$

  $$s_\mu [s_\nu] = \sum_\lambda a_\lambda(\mu, \nu) s_\lambda$$

describe **Schur functors** of irreducible $S_n$-reps
crucial in **Geometric Complexity Theory**

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Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley\(^\dagger\) (2000)

- **Schubert coefficients** $c(u, v, w) \in \mathbb{N}$

  $$\mathcal{G}_u \cdot \mathcal{G}_v = \sum_w c(u, v, w) \mathcal{G}_w$$

describe **cohomology of the Grassmannian**
Why work on combinatorial interpretations?

When you ask the experts, they tell you:

1) Intellectual curiosity
2) Need to publish
3) Blind belief in the mission
4) Getting estimates
5) Saturation-type problems (after Knutson-Tao)
6) Vanishing problems

**Estimate struggles:**

\[ 1 \leq g(\delta_k, \delta_k, \delta_k) \leq f^{\delta_k} = \sqrt{n!} e^{-O(n)} \]

where \( \delta_k = (k-1, \ldots, 2, 1) \), \( n = \binom{k}{2} \), \( f^{\delta_k} := \text{SYT}(\delta_k) \)

[Bessenrodt-Behns’04], [P.-Panova-Vallejo’16], [P.-Panova’20]

**Saturation struggles:**

Saturation easily fails for Kronecker coefficients, e.g.

\[ g(2^2, 2^2, 2^2) = 1 \quad \text{but} \quad g(1^2, 1^2, 1^2) = 0. \]

Moreover, saturation fails for the reduced Kronecker coefficients [P.-Panova’20]

Vanishing struggles:

Deciding if \( g(\lambda, \mu, \nu) > 0 \) is strongly NP-hard

[Ikenmeyer-Mulmuley-Walter’17]
Why would they exist?

**Positive experience**

1. **Young’s rule:**  \( f^\lambda = |\text{SYT}(\lambda)| \), where  \( f^\lambda := \chi^\lambda(1) \)  \[\text{Young, 1900}\]

2. **Littlewood–Richardson’s rule:**  \( c^\lambda_{\mu \nu} = |\text{LR}(\lambda/\mu, \nu)| \)  \[\text{L–R, 1934}\]

3. **Pipe dreams rule:**  \( \mathcal{G}_w = \sum_D x^D \)  \[\text{Fomin-Kirillov’96}, \text{Bergeron-Billey’93}, \text{Knutson-Miller’05}\]

4. (few/several/many) more extensions/generalizations/variations on the theme  \( \text{(many papers)} \)

**Perseverance & Optimism**  \( \text{(as in “why be discouraged by failures?”)} \)

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**Combinatorics Seminar**

**Thursday February 07, 2013**

- **Sami Assaf** (USC)
  - Stable Schur functions

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**Combinatorics Seminar**

**Thursday May 11, 2017**

- **Sami Assaf** (USC)
  - Schubert polynomials and slide polynomials

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**Kroneckers real soon!**

**Schuberts real soon!**
Why would they NOT exist?

Very brief history of negative results:

1. **rational numbers** $<$ **geometric numbers**
   
   *Pythagoras* (6th century BC): $\sqrt{2} \notin \mathbb{Q}$

2. **ruler/compass numbers** $<$ **geometric numbers**
   
   *Gauss* (1796): regular 7-gon cannot be constructed

3. **radical numbers** $<$ **algebraic numbers**
   
   *Ruffini* (1799), *Abel* (1824): quintic equations cannot be solved in the radicals

4. **elementary functions** $<$ **their integrals**
   
   *Liouville* (1833-41): $\int \frac{\sin x}{x} \, dx$, $\int e^{-x^2} \, dx$ are not elementary

5. **algebraic numbers** $<$ **reals**
   
   *Liouville* (1844), *Cantor* (1874): $\exists$ (many) transcendental numbers
Seriously, why *would* Kronecker coefficients have a combinatorial interpretation?

*Imaginary conversation of 15 y.o. Gauss and his friend:*

**Friend:** Why do you believe that the heptagon cannot be constructed?

**Gauss:** IDK. Because many smart people tried and failed. Why do you believe that it can?

**Friend:** Isn’t it obvious? We can construct so much: triangle, square, pentagon, hexagon, even octagon. I am very optimistic!

(Braunschweig, Germany, 1793)
What is a combinatorial interpretation?

Wrong Answer: *anything that we can count!*

**Cherednik (2002):**

*Combinatorics is the science of counting the possible arrangements and ways of organizing collections of anything (e.g., atoms, pebbles, star clusters).*

**Billey (Feb. 2021):** *I never say “It is an open problem to find a combinatorial interpretation for the Schubert coeff.” They already count something!*

**Billey to P. (Apr. 2022):** *They count the number of points in a generic intersection of 3 Schubert varieties.*
What is a combinatorial interpretation?

Wrong Answer: Popper: *A belief needs to be disprovable in order to be scientific!*

We need a formal definition!
What is a combinatorial interpretation?

Correct Answer: #P (a notion in computational complexity)
What is #P?

Quick and easy guide with examples:

(0) \( P \) – class of poly-time decision problems
\[ FP \] – class of poly-time counting problems

Examples: GraphConnectivity, PerfectMatching, \( c_{\mu\nu}^{\lambda} > 0 \in P \)
\#SpaningTrees, \#PerfectMatching in planar graphs, \( f^{\lambda}, s_\lambda(1,\ldots,1) \in FP \)

(1) \( NP \) – class of decision problems where objects can be verified in poly-time
\( NP \)-complete – class of hardest problems in \( NP \)

Examples: 3Coloring, HC (Hamiltonian cycle), Knapsack \( \in NP\text{-c} \)

(2) \#P \) – class of counting problems where objects can be verified in poly-time
\#P-complete – class of hardest problems in \#P

Examples: \#3Coloring, \#HC, \#Knapsack, \#PerfectMatching, \( c_{\mu\nu}^{\lambda} \in \#P\text{-c} \)
Where are our favorite problems?

(3) $\text{GapP} := \#P - \#P$ and $\text{GapP}_{\geq 0} := \text{GapP} \cap \mathbb{N}$

g(\lambda, \mu, \nu) \in \text{GapP}_{\geq 0} \quad \text{[Christandl-Doran-Walter’12], [P.-Panova’17]}

a_{\lambda}(\mu, \nu) \in \text{GapP}_{\geq 0} \quad \text{[Fischer-Ikenmeyer’20]}

c(u, v, w) \in \text{GapP}_{\geq 0} \quad \text{follows from [Postnikov-Stanley’09]}

Translation to Algebraic Combinatorics lingo:

$\text{GapP}_{\geq 0} = \text{“Combinatorial Interpretation”}$

$\#P = \text{“Manifestly Positive Combinatorial Interpretation”}$

Note: Billey’s “combinatorial interpretation” is not in $\#P$ because of Vakil’s Murphy’s law (2006)
Other problems in \( \text{GAPP}_{\geq 0} \)?

1. \( e(P) - 1 \in \text{GAPP}_{\geq 0} \)
   \[ P = (X, \prec) \text{ is a poset, } e(P) = \# \text{ linear extensions of } P \]

2. \( m_k(G)^2 - m_{k+1}(G)m_{k-1}(G') \in \text{GAPP}_{\geq 0} \)
   \[ m_k(G) := \# k\text{-matchings in } G \quad \text{[Heilmann-Lieb’72]} \]

3. \( (2m)^{n-1} - n(n - 1)^{n-1}\tau(G') \in \text{GAPP}_{\geq 0} \)
   \[ G = (V, E), |V| = n, |E| = m, \tau(G) = \# \text{ spanning trees} \quad \text{[Grimmett’76]} \]

4. \( f_k(G)^2 - f_{k+1}(G)f_{k-1}(G') \in \text{GAPP}_{\geq 0} \)
   \[ f_k(G') := \# k\text{-forests in } G \quad \text{[Adiprasito-Huh-Katz’18]} \]

\( (1) \) and \( (3) \) \( \in \#P \) (easy)

\( (2) \in \#P \) by [Krattenthaler’96]

\( (4) \in \#P \) is a major open problem. No combinatorial proof is known.

see also [Anari-Liu-Oveis.Gharan-Vinzant’18], [Brändén-Huh’20], [Chan-P.’21]
Can we prove anything at all?

Yes, now we can!

**Proposition** [Ikenmeyer-P.'22]

If \( \text{GapP}^2 \subseteq \#P \), then \( \text{PH} = \Sigma^P_2 \).

**Note:** \( \text{GapP}^2 = (\#P - \#P)^2 \subseteq \text{GapP}_{\geq 0} \)

**Explanation of Proposition:** Let \( G, H \) be two graphs. Define

\[
f(G, H) := \left( \#3\text{-colorings in } G - \#3\text{-colorings in } H \right)^2
\]

Then \( f(G, H) \notin \#P \), i.e. does not have a combinatorial interpretation unless (\(*\))

**Explanation of (\(*\)):** You can decide \( \exists \forall \exists \ldots \forall \Phi \) just as fast as \( \exists \forall \Phi \).

This is universally conjectured to be false (\( \text{PH} \neq \Sigma^P_2 \) is stronger than \( \text{P} \neq \text{NP} \)).

**Proof idea:** Suppose \( f(G, H) \in \#P \). Then there is a poly-time certificate for

\( (\#3\text{-colorings of } G) \neq (\#3\text{-colorings of } H) \). But that would be too powerful, akin to poly-time certificate for \( (\#3\text{-colorings of } G) = 0 \).
Our results (a sampler):

The following are $\not\in \#P$ (under some complexity assumptions)

**Cauchy inequality**
\[(x_1y_1 + \ldots + x_ny_n)^2 \leq (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2)\]

**Minkowski inequality**
\[\prod_{i=1}^{n}(x_i^n + y_i^n) \geq \left[ \prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i \right]^n\]

**Karamata inequality**

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, such that $\boldsymbol{x} \succeq \boldsymbol{y}$.

Then, for every convex $f : \mathbb{R}^n \to \mathbb{R}$, we have $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$.

**Definition:** Let $\boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be nonincreasing sequences.

We say: $\boldsymbol{x}$ majorizes $\boldsymbol{y}$, write $\boldsymbol{x} \succeq \boldsymbol{y}$, if
\[
x_1 + \ldots + x_i \geq y_1 + \ldots + y_i \quad \text{for all } 1 \leq i < n, \quad \text{and}
\]
\[
x_1 + \ldots + x_n = y_1 + \ldots + y_n.
\]

**Hadamard inequality**
\[
\det \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}^2 \leq \prod_{i=1}^{d} (a_{i1}^2 + \ldots + a_{id}^2)
\]
Case study: **Gessel sequence**

\[ b_n := 2 \cdot 5^n - (3 + 4i)^n - (3 - 4i)^n, \quad \text{where} \quad i = \sqrt{-1} \]

Note that \( b_n \in \mathbb{Z} \) since

\[ b_n = 2 \cdot 5^n - 2 \sum_r (-1)^r \binom{n}{2r} 3^{n-2r} 4^{2r} \]

and that \( b_i \geq 0 \) since \( |3 \pm 4i| = 5 \).

\[ b_n = [2 \text{Im}(1 + 2i)^n]^2 \]

\[ b_n = -b_{n-1} + 5b_{n-2} + 125b_{n-3} \quad \text{for} \quad n > 2. \]

\[ B(t) := \sum_{n=0}^{\infty} b_n t^n = \frac{16t(1 + 5t)}{(1 - 5t)(1 + 6t + 25t^2)} \quad \text{not} \quad \mathbb{N}\text{-rational} \]

**Open Problem:** Does \( \{b_n\} \) have a combinatorial interpretation?
Case study: *generalized Gessel sequences*

Consider \( \{a_n\} = \{a_n(f, g)\} \) defined as

\[
a_n := 2(f^2 + g^2)^n - (f + gi)^{2n} - (f - gi)^{2n}.
\]

\[
a_n(1, 2) = b_n
\]

\[
A(t) := \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Z}(t) \cap \mathbb{N}[[t]].
\]

Suppose now that \( f, g \) are \( \#P \) functions.

\[
a_1 = 4g^2 \in \#P
\]

\[
a_2 = 16f^2g^2 \in \#P
\]

\[
a_3 = 4g^2(3f^2 - g^2)^2 \notin \#P \text{ unless UP = coUP}
\]

\[
a_4 = [8fg(f + g)(f - g)]^2 \notin \#P \text{ unless PH = } \Sigma_2^P
\]

**Conjecture:** \( a_n(f, g) \notin \#P \) for all \( n \geq 3 \).
Thank you!