Cohomology of line bundles on flag varieties

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Reduced homology of the simplex

$$\mathbf{k} = \text{a field (or } \mathbb{Z}), \qquad d \geq 0 \text{ an integer, and write}$$

$$[d] = \{1, \cdots, d\}.$$

Consider the complex $C_\bullet = C_\bullet(d)$, with

$$C_t = \bigoplus_{J \in \binom{[d]}{t}} \mathbf{k} \cdot e_J \simeq \mathbf{k}^{\oplus \binom{d}{t}}, \quad t = 0, \cdots, d,$$

and differential

$$\partial(e_{j_1, \cdots, j_t}) = \sum (-1)^{i-1} e_{j_1, \cdots, \widehat{j_i}, \cdots, j_t}.$$

For $d = 3$, we get

$$0 \longrightarrow \mathbf{k} \xrightarrow{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}} \mathbf{k} \longrightarrow 0$$

Exercise. $C_\bullet$ is exact (for all $d > 0$ and all $\mathbf{k}$).

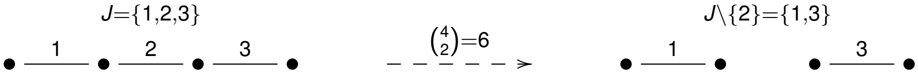
What if we rescale entries?

Exercise. Study how the homology depends on $\mathbf{k}$:

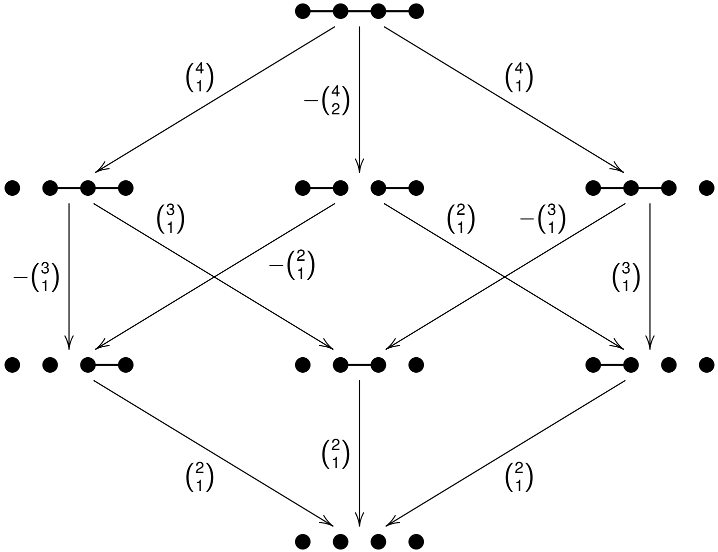
$$0 \longrightarrow \mathbf{k} \xrightarrow{\begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} -3 & -2 & 0 \\ 3 & 0 & -3 \\ 0 & 2 & 3 \end{bmatrix}} \mathbf{k}^3 \xrightarrow{\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}} \mathbf{k} \longrightarrow 0$$

To construct such a complex, think of the elements of $[d]$ as edge labels

- each $J \subseteq [d]$ gives a disjoint union of intervals.
- removing an element j from J breaks exactly one interval, of size ($:=$ number of vertices) w, into two intervals of size w' and $w - w'$.
- construct $\tilde{C}_\bullet = \tilde{C}_\bullet(d)$ from $C_\bullet$ by replacing ± 1 with $\pm\binom{w}{w'}$.



An arithmetic Koszul complex



Integers $\equiv 0, 1 \bmod p$

For a prime $p > 0$, enumerate non-negative integers $\equiv 0, 1 \bmod p$:

$$0, 1, p, p+1, 2p, 2p+1, \cdots$$

If m is in the list above, write $|m|_p$ for its position (p-index):

$$\text{if } m = pa + b, \text{ with } b \equiv 0, 1 \bmod p, \text{ then } |m|_p = 2a + b.$$

For $p = 3$:

m	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\|m\|_p$	0	1		2	3		4	5		6	7		$\cdots$

For a tuple $\alpha = (\alpha_0, \cdots, \alpha_k)$, with $\alpha_i \equiv 0, 1 \bmod p$, we write

$$|\alpha|_p = \sum_{i=0}^{k} |\alpha_i|_p, \text{ and let}$$

$$A_{p,d} = \{\alpha = (\alpha_0, \cdots, \alpha_k) : \sum \alpha_i \cdot p^i = d, \ \alpha_i \equiv 0, 1 \bmod p\}.$$

Examples: $A_{3,8} = \emptyset$, $A_{3,9} = \{(0,0,1), (0,3,0), (6,1,0), (9,0,0)\}$.

$$|(0,0,1)|_3 = 1, \ |(0,3,0)|_3 = 2, \ |(0,6,1)|_3 = 5, \ |(9,0,0)|_3 = 6.$$

The homology of $\tilde{C}_\bullet$ in characteristic $p > 0$

Theorem (R–VandeBogert)

Suppose that $\mathrm{char}(\mathbf{k}) = p > 0$*, and write*

$$P_d(t) := \sum_{i \geq 0} \dim_{\mathbf{k}} H_i(\tilde{C}_\bullet(d)) \cdot t^i.$$

1. $P_d(t) = \sum_{\alpha \in A_{p,d+1}} t^{d+1-|\alpha|_p}$.
2. *Consider the projective space* $\mathbf{P}^r$ *over* $\mathbf{k}$*, with* $r > d$*. If we write* Ω *for the cotangent bundle on* $\mathbf{P}^r$*, then*

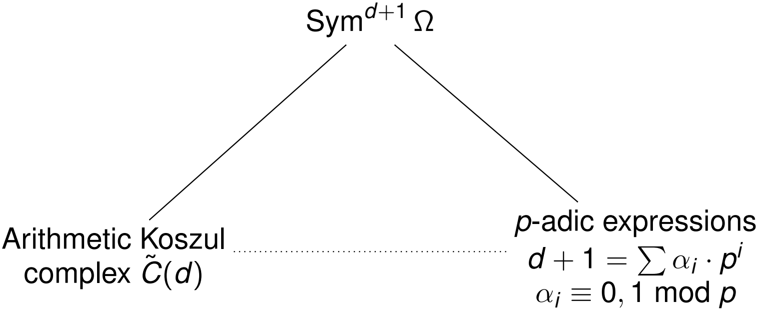
$$\sum_{i \geq 0} \dim_{\mathbf{k}} H^i\left(\mathbf{P}^r, \mathrm{Sym}^{d+1}\Omega\right) \cdot t^i = t^{d+1} \cdot P_d(t^{-1}) = \sum_{\alpha \in A_{p,d+1}} t^{|\alpha|_p}.$$

$d = 0$: $P_0(t) = 1$ and $H^1(\mathbf{P}^r, \Omega) = \mathbf{k}$.

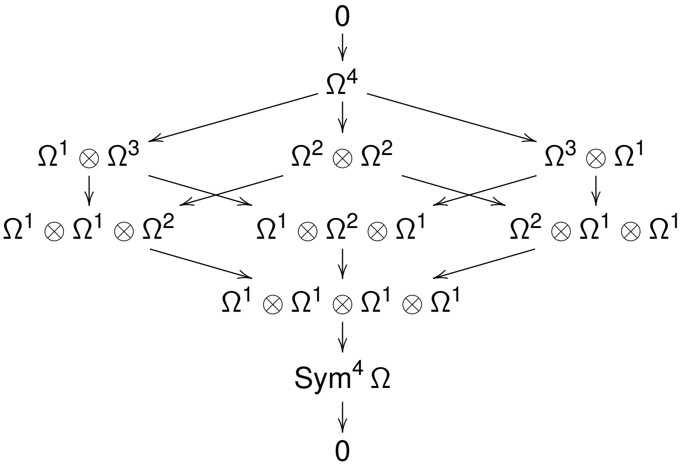
$d = 1$, $p = 2$: $P_1(t) = 1 + t$ and $H^i(\mathbf{P}^r, \mathrm{Sym}^2\Omega) = \mathbf{k}$ for $i = 1, 2$.

$d = 8$, $p = 3$: $P_9(t) = t^{9-6} + t^{9-5} + t^{9-2} + t^{9-1} = t^3 + t^4 + t^7 + t^8$.

Proof outline



Resolution by exterior powers



Akin,
Buchsbaum,
Rota,
Totaro,
. . . .
resolutions of
$\mathrm{GL}(V)$-modules

- $H^{i_1+\cdots+i_k}\left(\mathbf{P}^r, \Omega^{i_1} \otimes \cdots \otimes \Omega^{i_k}\right) = \mathbf{k}$, and $H^j = 0$ for $j \neq i_1 + \cdots + i_k$.
- $\Omega^{i_1} \otimes \Omega^{i_2} \longrightarrow \Omega^{i_1+i_2}$ induces an isomorphism in cohomology.
- $\Omega^{i_1+i_2} \longrightarrow \Omega^{i_1} \otimes \Omega^{i_2}$ is multiplication by $\binom{i_1+i_2}{i_1}$ in cohomology.

Truncated symmetric powers

Let V be a $\mathbf{k}$-vector space, $\dim(V) = n$, $\mathrm{char}(\mathbf{k}) = p > 0$.

$$S = \mathrm{Sym}(V) \simeq \mathbf{k}[x_1, \cdots, x_n], \text{ the symmetric algebra of } V.$$

Consider the Frobenius powers

$$F^p V = \langle v^p : v \in V \rangle_{\mathbf{k}} \subset \mathrm{Sym}^p V,$$

and the truncated symmetric algebra

$$T_p S = \mathrm{Sym}(V)/\langle F^p V \rangle \simeq \mathbf{k}[x_1, \cdots, x_n]/\langle x_1^p, \cdots, x_n^p \rangle.$$

For $\alpha = (\alpha_0, \cdots, \alpha_k)$, write

$$F^\alpha V = T_p \mathrm{Sym}^{\alpha_0} V \otimes F^p(T_p \mathrm{Sym}^{\alpha_1} V) \otimes \cdots \otimes F^{p^k}(T_p \mathrm{Sym}^{\alpha_k} V).$$

Doty: the composition factors of $\mathrm{Sym}^d V$ are $F^\alpha V$, with $\sum \alpha_i \cdot p^i = d$.

R–VandeBogert: the only non-zero cohomology for $F^\alpha \Omega$ is $H^{|\alpha|_p}(F^\alpha \Omega) = \mathbf{k}$, and it occurs if and only if $\alpha_i \equiv 0, 1 \bmod p$ for all i.

(Partial) flag varieties

Let $V \simeq \mathbf{k}^n$ be a vector space of dimension n.

$\mathrm{Flag}(V)$ = the flag variety parametrizing complete flags of subspaces

$$V_\bullet : \quad V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V, \text{ where } \dim(V_i) = i.$$

For a subset $J = \{j_1, \cdots, j_k\} \subset \{1, \cdots, n-1\}$, the partial flag variety

$$\mathrm{Flag}(J, V)$$

parametrizes partial flags of subspaces

$$V_{j_1} \subset V_{j_2} \subset \cdots \subset V_{j_k} \subset V, \text{ where } \dim(V_{j_i}) = j_i.$$

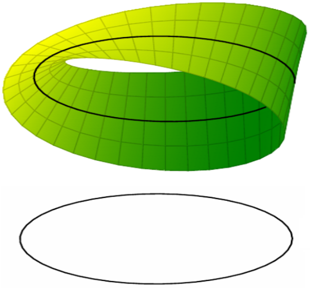
Examples:

- Grassmannians (when $J = \{k\}$, $1 \leq k \leq n-1$);
- projective spaces ($J = \{1\}$ or $J = \{n-1\}$).
- incidence correspondence ($J = \{1, n-1\}$).

(Tautological) line and vector bundles

On $\mathrm{Flag}(V)$, we have:

- $\mathcal{U}_i$ = tautological sub bundle, with fiber V_i at $[V_\bullet]$.
- $\mathcal{Q}_i$ = tautological quotient bundle, with fiber V/V_{n-i} at $[V_\bullet]$.
- $\mathcal{L}_i = \ker(\mathcal{Q}_i \twoheadrightarrow \mathcal{Q}_{i-1})$ tautological line bundle.



There are similar bundles on $\mathrm{Flag}(J, V)$.

Open Problem

Determine the sheaf cohomology groups ($H^0, H^1, H^2, \cdots$) for every line bundle on $\mathrm{Flag}(V)$ *or* $\mathrm{Flag}(J, V)$*.*

Open Problem'

Determine which of the sheaf cohomology groups are zero, and which ones are non-zero.

Classification of line bundles

The Picard group for Flag(V) is

$$\mathrm{Pic}(\mathrm{Flag}(V)) \simeq \frac{\mathbb{Z}^n}{\mathbb{Z}\cdot(1,\cdots,1)},$$

generated by $\mathcal{L}_1,\cdots,\mathcal{L}_n$ with relation $\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n \simeq \mathcal{O}_{\mathrm{Flag}(V)}$. Write

$$\mathcal{O}(\lambda) = \mathcal{L}_1^{\lambda_1}\otimes\mathcal{L}_2^{\lambda_2}\otimes\cdots\otimes\mathcal{L}_n^{\lambda_n}.$$

Given a partial flag variety Flag(J, V), there is a forgetful map

$$f:\mathrm{Flag}(V)\longrightarrow\mathrm{Flag}(J,V),\ \text{and}$$

1. $f^*:\mathrm{Pic}(\mathrm{Flag}(J,V))\longrightarrow\mathrm{Pic}(\mathrm{Flag}(V))$ is injective, and

   $$H^j(\mathrm{Flag}(J,V),\mathcal{L}) = H^j(\mathrm{Flag}(V),f^*\mathcal{L}),$$

   where $f^*\mathcal{L}=\mathcal{O}(\lambda)$, with certain consecutive λ_i equal to each other.
2. Often $f_*(\mathcal{L})$ is a vector bundle with the same cohomology as $\mathcal{L}$. E.g., if $f:\mathrm{Flag}(V)\longrightarrow\mathbb{P}V$ and $\lambda=(-d-1,d+1,0,\cdots,0)$ then $f_*(\mathcal{O}(\lambda))=\mathrm{Sym}^{d+1}\,\Omega$ has the same cohomology as $\mathcal{O}(\lambda)$.

Effective cone and Kempf vanishing

The effective cone (of line bundles $\mathcal{L}$ with $H^0(\mathcal{L}) \neq 0$) is spanned by the fundamental weights

$$\omega_i = (1, \cdots, 1, 0, \cdots, 0) \longleftrightarrow \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_i = \det(\mathcal{Q}_i).$$

In other words,

$$H^0(\mathcal{O}(\lambda)) \neq 0 \Longleftrightarrow \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

in which case we say λ is dominant. In this case, one has

$$H^0(\mathcal{O}(\lambda)) = \mathbb{S}_\lambda V,$$

the Schur functor associated to λ.

Theorem (Kempf '76, Haboush '80, Andersen '80)

If λ is dominant, then

$$H^i(\mathcal{O}(\lambda)) = 0 \text{ for all } i > 0.$$

The Borel–Weil–Bott theorem

Theorem (Borel–Weil–Bott)

Suppose that $\operatorname{char}(\mathbf{k}) = 0$*, and let* $\lambda \in \mathbb{Z}^n/\mathbb{Z} \cdot (1, \cdots, 1)$*.*

(a) *There exists at most one value of i such that* $H^i(\mathcal{O}(\lambda)) \neq 0$*.*

(b) *If* $\lambda_i - i = \lambda_j - j$ *for some* $i \neq j$*, then*

$$H^i(\mathcal{O}(\lambda)) = 0 \text{ for all } i.$$

(c) *When* $H^i(\mathcal{O}(\lambda)) \neq 0$*, it is an irreducible* SL_n*-representation.*

Example: If $r > d$ then $H^i\left(\mathbf{P}^r, \operatorname{Sym}^{d+1}\Omega\right) = 0$ for all i.

Proof: take $n = r + 1$, so that $\mathbb{P}V \simeq \mathbf{P}^r$, and $\lambda = (\lambda_1, \cdots, \lambda_{r+1})$, where

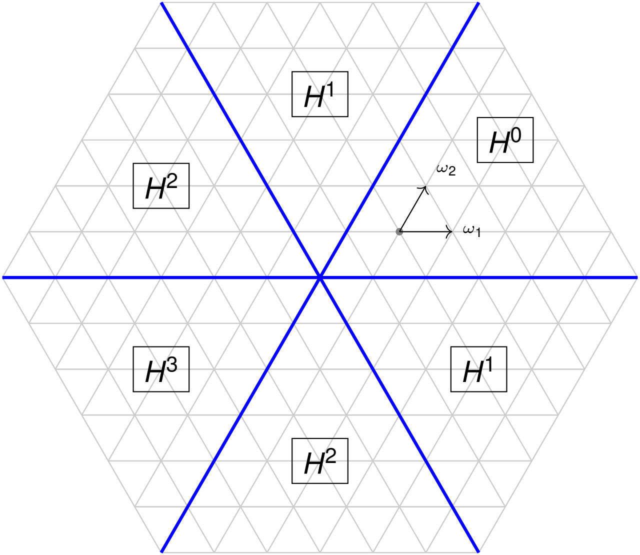
$$\lambda_1 = -d - 1,\ \lambda_2 = d + 1, \lambda_i = 0 \text{ for } i > 2.$$

Note: $\lambda_1 - 1 = \lambda_{d+2} - (d + 2)$ (which exists, since $d + 2 \leq r + 1$).

$$H^i\left(\mathbf{P}^r, \operatorname{Sym}^{d+1}\Omega\right) = H^i\left(\mathcal{O}(\lambda)\right) \overset{(b)}{=} 0 \quad \text{for all } i.$$

Borel–Weil–Bott for $\mathrm{Flag}(\mathbf{k}^3) = \mathrm{SL}_3(\mathbf{k})/B$

Let $\omega_1 = (1, 0, 0)$ and $\omega_2 = (1, 1, 0)$ in $\mathbb{Z}^3/\mathbb{Z}(1, 1, 1)$.



How about $\mathrm{char}(\mathbf{k}) = p > 0$?

Example (Mumford). If $p = 2$, $n = 3$, $\lambda = (-2, 2, 0)$, then

$$H^1(\mathcal{O}(\lambda)) = H^2(\mathcal{O}(\lambda)) = \mathbf{k}.$$

Andersen '79: characterization of the weights λ for which

$$H^1(\mathcal{O}(\lambda)) \neq 0.$$

By Serre duality, get characterizations of when:

- $H^{\binom{n}{2}}(\mathcal{O}(\lambda)) \neq 0$ (using Kempf vanishing).
- $H^{\binom{n}{2}-1}(\mathcal{O}(\lambda)) \neq 0$ (using Andersen '79).

For $n = 3$:

- **Griffith '80:** characterizes (non-)vanishing of $H^i(\mathcal{O}(\lambda))$.
- **Donkin '06:** (complicated) recursive formula for $H^i(\mathcal{O}(\lambda))$.
- **Liu '19:** vastly simplified recursion for $H^i(\mathcal{O}(\lambda))$.

Schur functors of the cotangent sheaf on $\mathbf{P}^r$

Open Problem

Describe / give a recursive formula for

$$H^i(\mathbf{P}^r, \mathbb{S}_\mu \Omega),$$

where $\mu = (\mu_1 \geq \cdots \geq \mu_r \geq 0)$.

Equivalently, describe the cohomology of $\mathcal{O}(\lambda)$ on $\mathrm{Flag}(\mathbf{k}^{r+1})$, where

$$\lambda = (-|\mu|, \mu_1, \mu_2, \cdots, \mu_r).$$

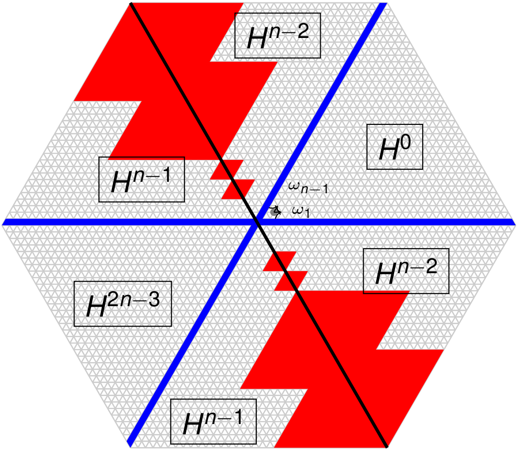
R–VandeBogert:

- If $r \gg 0$ then each $H^i(\mathbf{P}^r, \mathbb{S}_\mu \Omega)$ has a trivial SL-action.
- Can describe recursively what happens when μ is a hook partition, or μ is a two column partition ($\mu_1 \leq 2$).

Cohomology of line bundles on Flag($\{1, n-1\}, V$)

Theorem (Gao–R)

Compared to the characteristic zero cohomology, if $\mathrm{char}(\mathbf{k}) = p > 0$ *then the line bundles in the red region have additional cohomology, distributed evenly between degrees* $(n-2)$ *and* $(n-1)$*:*



- groups of $(p-1)$ triangles.
- symmetries: $V \longleftrightarrow V^\vee$ and Serre duality.
- equivalent to cohomology of $\mathrm{D}^d\,\Omega(e)$ (twists of divided powers) on $\mathbb{P}V$.
- non-vanishing by work of Andersen (Frobenius splittings).
- vanishing by analyzing Castelnuovo–Mumford regularity for $\mathrm{D}^d\,\Omega$.

Truncated Schur polynomials

Write $[M]$ for the character of an SL-module M, we have

$$[\mathrm{Sym}^d V] = h_d, \text{ the } d\text{-th complete symmetric polynomial, and}$$

$$[\mathbb{S}_\lambda V] = \mathcal{S}_\lambda = \det(h_{\lambda_i+j-i}), \text{ the Schur polynomial.}$$

Define similarly truncated symmetric polynomials

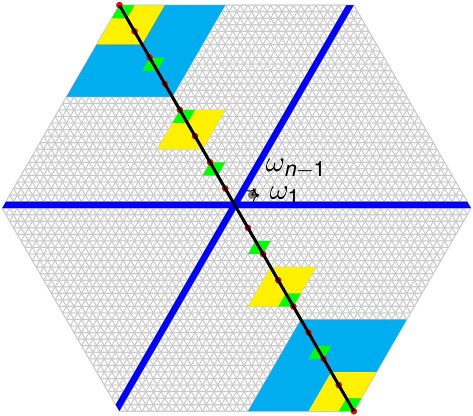
$$h_d^{(p)} := [T_p \,\mathrm{Sym}^d V],$$

$$\mathcal{S}_\lambda^{(p)} := \det(h^{(p)}_{\lambda_i+j-i}),$$

as well as analogues $h_d^{(q)}$, $\mathcal{S}_\lambda^{(q)}$ when $q = p^k$.

Caution! $\mathcal{S}_\lambda^{(p)}$ is usually only a virtual character.

Cohomology layers for Flag($\{1, n-1\}, V$)



Conjectural:

- extra cohomology decomposes into layers.
- building blocks come from $\mathcal{S}^{(q)}_{(e+q,d-q)}$, where $q = p^k$.
- if $p = 2$, the multiplicity of the layers described by Nim symmetric polynomials.

More problems

Open Problem

Describe the cohomology of line bundles on $\mathrm{Flag}(\{1,2\}, V)$*, that is, for line bundles* $\mathcal{O}(\lambda)$ *with* $\lambda = (\lambda_1, \lambda_2, 0, \cdots, 0)$*. Or, describe the cohomology of* $(\mathrm{Sym}^d\,\Omega)(e)$ *on* $\mathbb{P}V$*, for all* d, e*.*

Open Problem

For which λ *is the truncated Schur polynomial* $\mathcal{S}_\lambda^{(p)}$ *an "honest" (non-virtual) character? What are its simple composition factors? Is there a "natural" realization, and a "nice" basis?*

Conjecture

If $a - b \geq p - 1$*, then every simple composition factor* $L(\mu)$ *of* $T_p \mathrm{Sym}^a V \otimes T_p \mathrm{Sym}^b V$ *has the property that* μ *is p-restricted, that is,*

$$\mu = \sum a_i \omega_i, \quad 0 \leq a_i < p.$$

. . . .

Thank You!