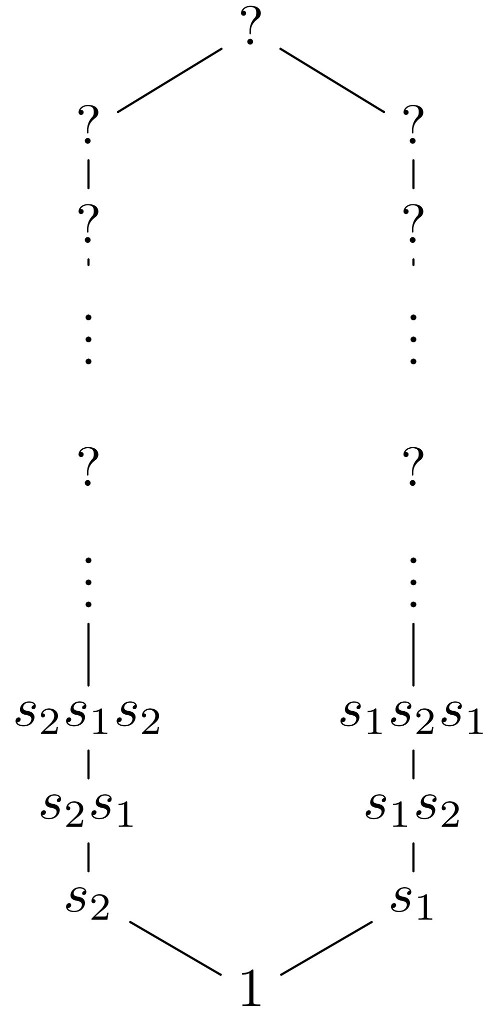
Extending weak order

David E Speyer

Slides at `http://www.math.lsa.umich.edu/~speyer/OPACSlides3.pdf`



Agenda

- **What** is a Coxeter group?
- **What** is the weak order on a Coxeter group?
- **Why** should it be enlarged?
- **How** could it be enlarged: Geometry of roots
    - Matthew Dyer's conjectures
    - Work with Grant Barkley
- **How** could it be enlarged: Lattice theory
    - Nathan Reading's theory of shards
    - Work with Nathan Reading and Hugh Thomas

Coxeter groups

A ***Coxeter group*** is generated by s_1, s_2, …, s_r modulo relations:

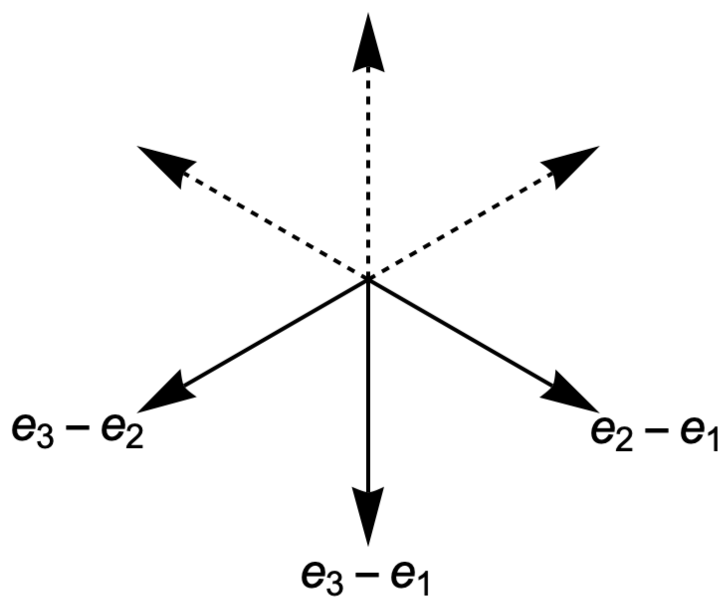
$$\begin{aligned} s_i^2 &= 1 \\ (s_i s_j)^{m_{ij}} &= 1 \quad \text{for some } 2 \leq m_{ij} \leq \infty \end{aligned}$$

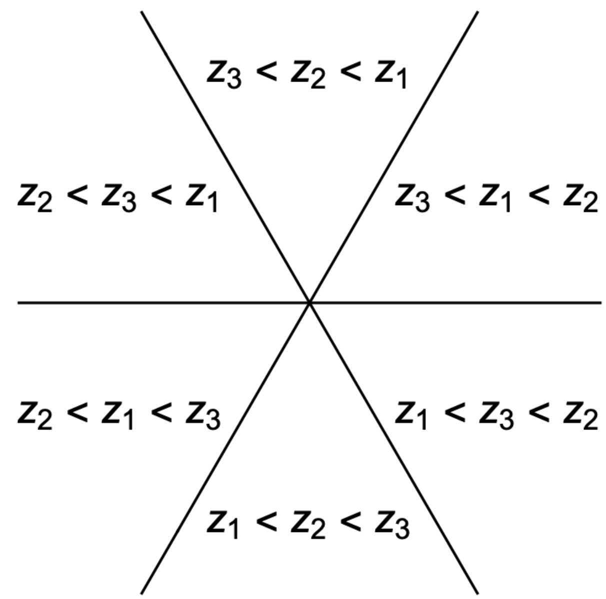
Coxeter groups come with linear representations: There is a vector space V with basis α_1, α_2, …, α_r such that the Coxeter group acts on V by a reflection, negating α_i. Equivalently, W acts on $V^\vee$ by a reflection fixing $\alpha_i^\perp$.

The symmetric group S_n is generated by s_1, s_2, $\ldots$, s_{n-1} where $s_i = (i\ i+1)$.

The symmetric group S_n is generated by s_1, s_2, ..., s_{n-1} where $s_i = (i\ i+1)$.

Roughly, take $V = V^\vee = \mathbb{R}^n$ and α_i is $e_{i+1} - e_i$.





More carefully, $V = (1, 1, \ldots, 1)^\perp \subset \mathbb{R}^n$ and $V^\vee = \mathbb{R}^n/\mathbb{R}(1, 1, \ldots, 1)$.

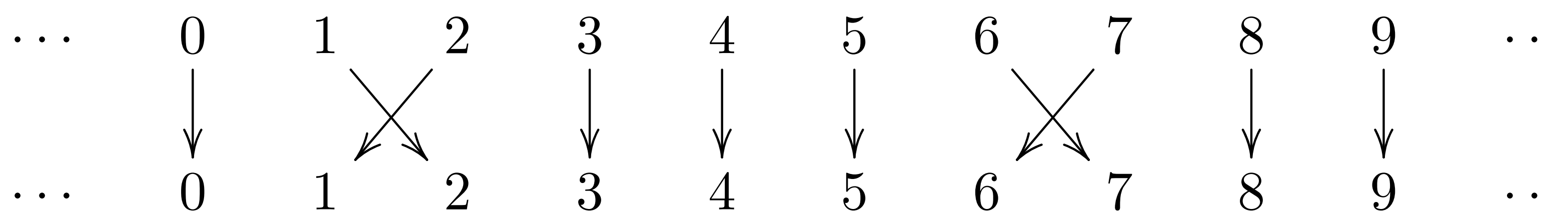
The affine symmetric group is the group of bijections $f : \mathbb{Z} \to \mathbb{Z}$ obeying

$$f(a+n) = f(a) + n \qquad \sum_{i=1}^{n} \big(f(a) - a\big) = 0.$$

The generators are

$$s_i = \cdots (i \ i+1)(i+n \ i+n+1)(i+2n \ i+2n+1) \cdots .$$

Here is s_1, with $n = 5$:



Let $V_1^\vee$ be the $(n+1)$-dimensional vector space of infinite real sequences $(\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots)$ such that there is a constant d with $z_{i+n} = z_i + d$ for all i. The affine symmetric group acts by permuting the subscripts.

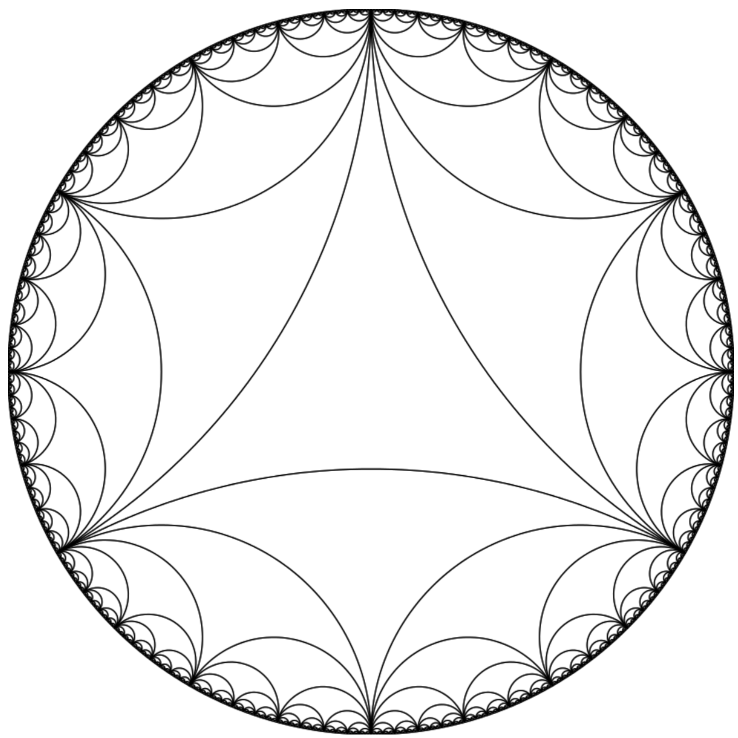
For $a \in \mathbb{Z}$, let $e_a : V_1^\vee \to \mathbb{R}$ be "evaluation at a"; these are vectors in the dual space V_1. So we have $e_{a+n} - e_a = e_{b+n} - e_b$ for all a, $b \in \mathbb{Z}$. Put $\delta = e_{a+n} - e_a$. So e_1, e_2, $\ldots$, e_n, δ is a basis of V_1.

Our α_i is $e_{i+1} - e_i$.

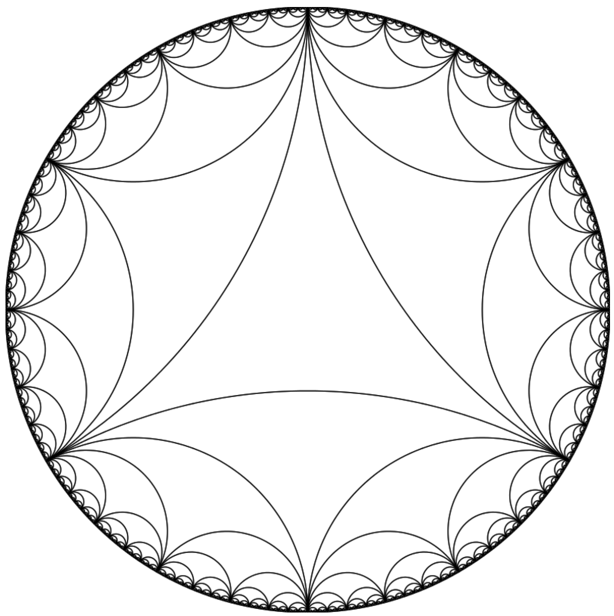
To be careful, V is the subset of V_1 spanned by the $e_i - e_j$; it's dual is $V_1^\vee$ modulo the constant sequences.

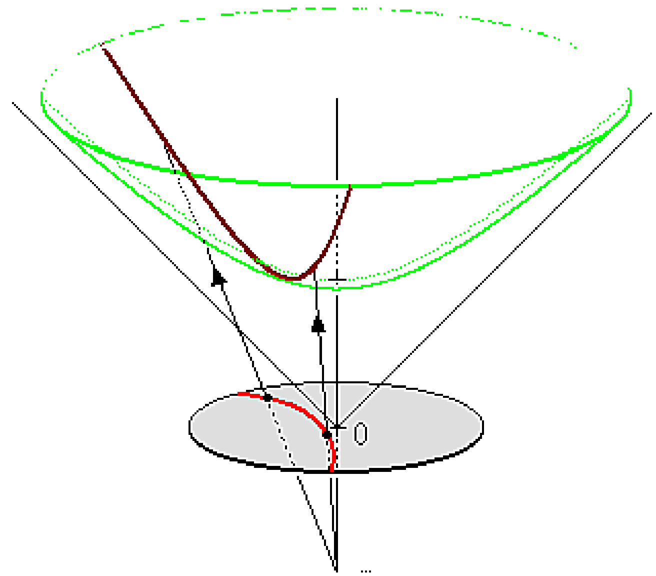
• **The free Coxeter group of rank** 3 has $m_{12} = m_{13} = m_{23} = \infty$. So it is generated by s_1, s_2 and s_3 modulo the relations $s_1^2 = s_2^2 = s_3^2 = 1$.

Geometrically, we can think of the group of symmetries of hyperbolic plane generated by reflections over three lines which meet at infinity.



If we use the “hyperboloid” model for hyperbolic space, we can think of this as symmetries of $\mathbb{R}^3$ preserving a quadratic form with signature $++-$.



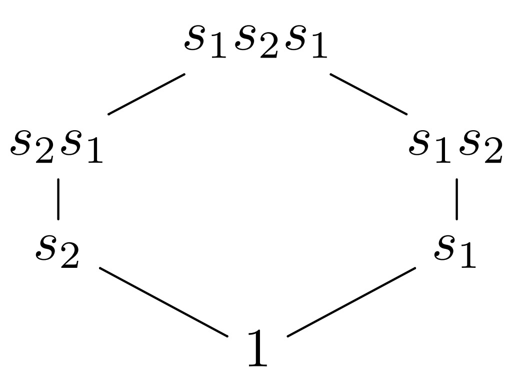


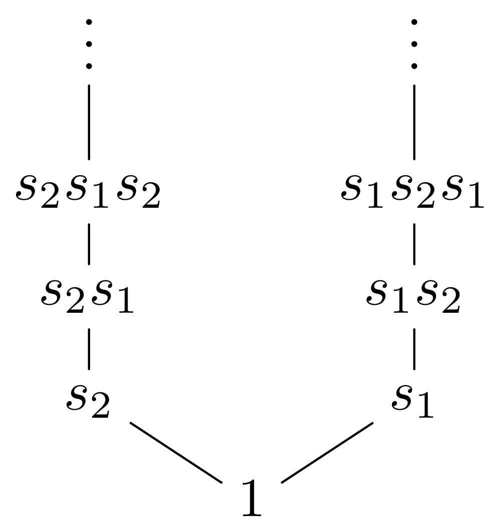
The simple roots α_1, α_2, α_3 are “space-like” vectors, sticking out to the side of the lightcone.

Weak order

A word $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ in the s_i is called ***reduced*** if it is of minimal length among words given this product.

The ***weak order*** is the partial order where $u \leq v$ if there is a reduced word $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for v with a prefix $s_{i_1} s_{i_2} \cdots s_{i_k}$ with product u.





We can give a more geometric description using the ideas of root systems and inversions.

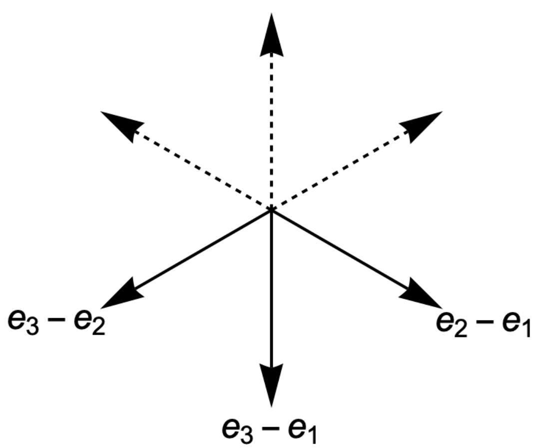
Roots and inversions

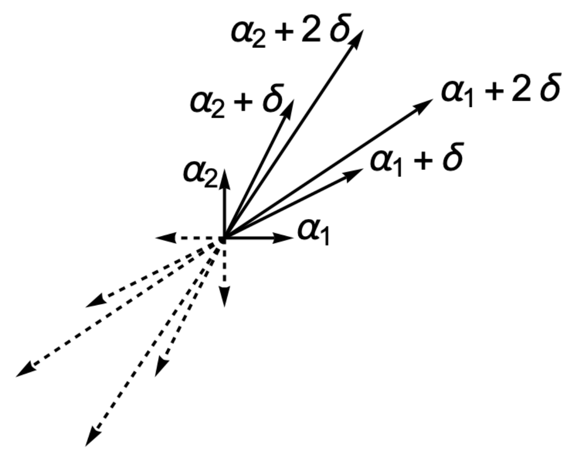
Let $\Phi = \bigcup_{w \in W} \{w\alpha_1, w\alpha_2, \ldots, w\alpha_n\}$. This is the ***root system***.

Every root is either a ***positive root***, meaning in $\mathbb{R}_{\geq 0}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, or a ***negative root***, meaning in $\mathbb{R}_{\leq 0}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. So $\Phi = \Phi^+ \sqcup \Phi^-$.

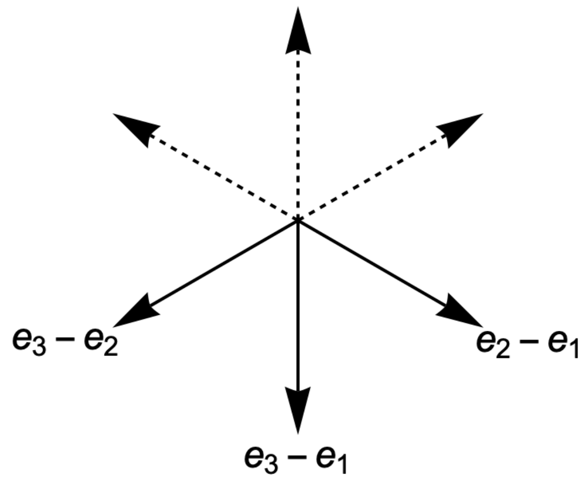
In the symmetric group, the positive roots are $e_j - e_i$ for $1 \leq i < j \leq n$.

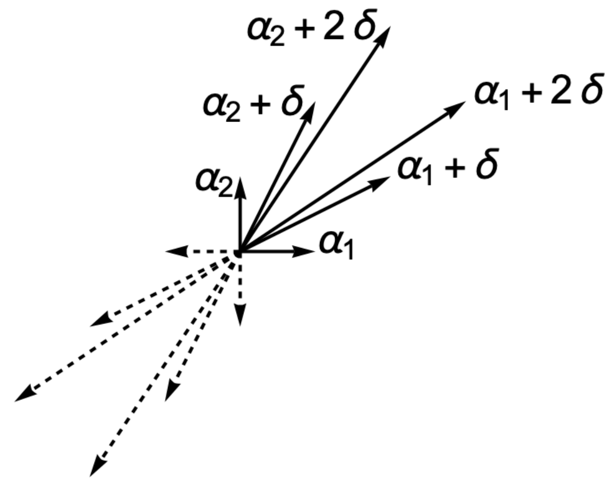
In the affine symmetric group, the positive roots are $e_j - e_i$ for $i < j$, $i \not\equiv j \bmod n$. Recall that $e_{i+n} - e_i = \delta$.

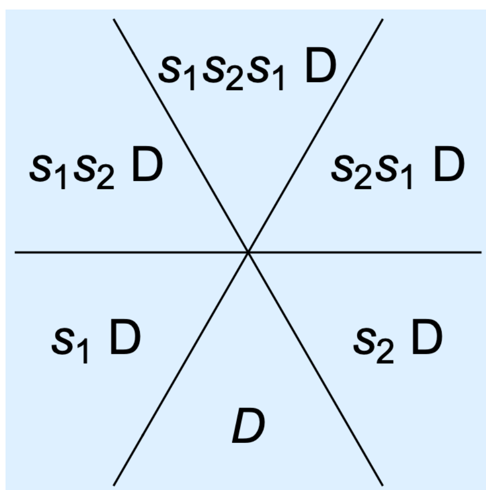


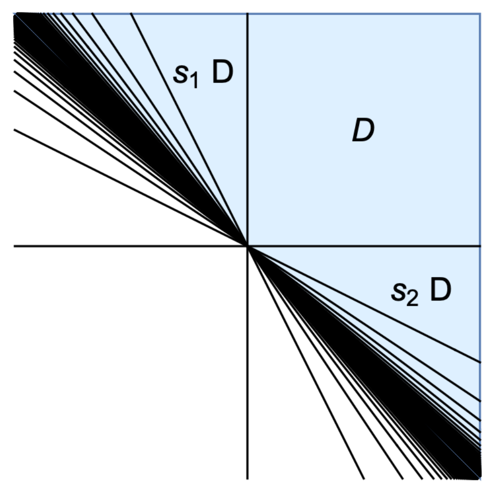


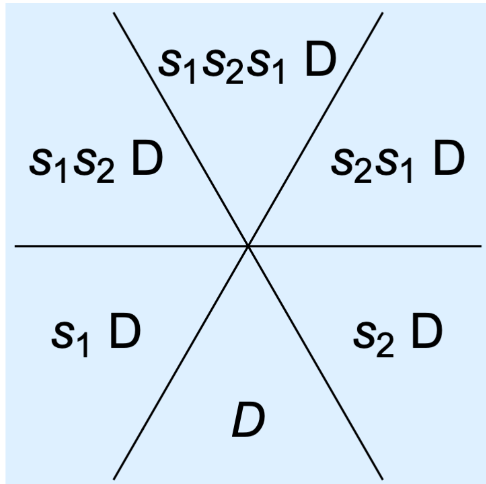
Let $V^\vee$ be the dual vector space to V. Each $\beta \in \Phi^+$ defines a dual hyperplane $\beta^\perp$ in $V^\vee$. Let $D = \{x \in V^\vee : \langle \alpha_i, x \rangle > 0 \text{ for } 1 \leq i \leq n\}$.

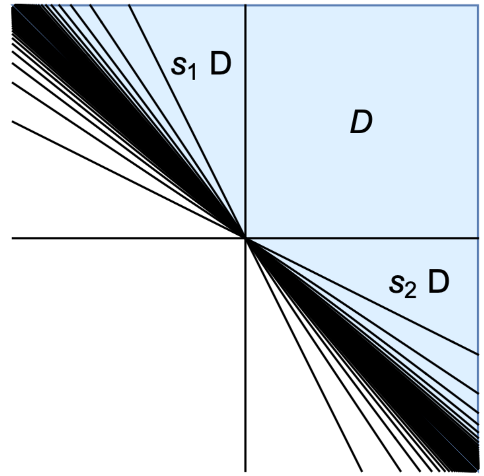




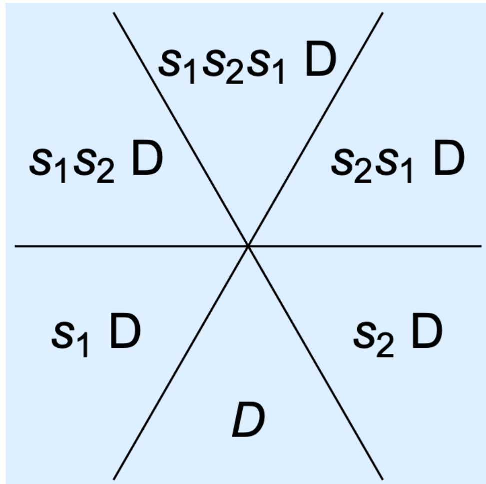


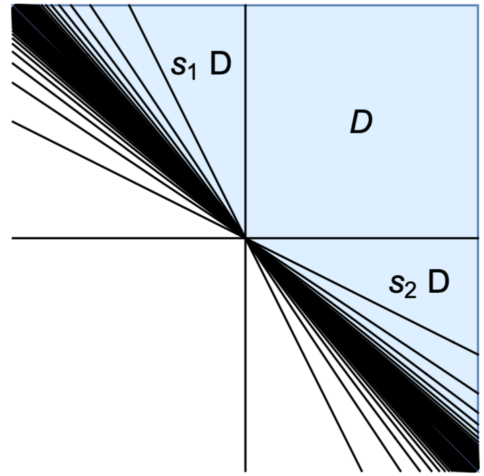






Theorem The uD, for $u \in W$, are always disjoint open simplicial cones. In finite type, they are the regions of the complement of the hyperplane arrangement. In general, they are precisely the regions where $\langle \beta, \ \rangle$ is positive for all but finitely many $\beta \in \Phi^+$.





Theorem The uD, for $u \in W$, are always disjoint open simplicial cones. In finite type, they are the regions of the complement of the hyperplane arrangement. In general, they are precisely the regions where $\langle \beta, \ \rangle$ is positive for all but finitely many $\beta \in \Phi^+$.

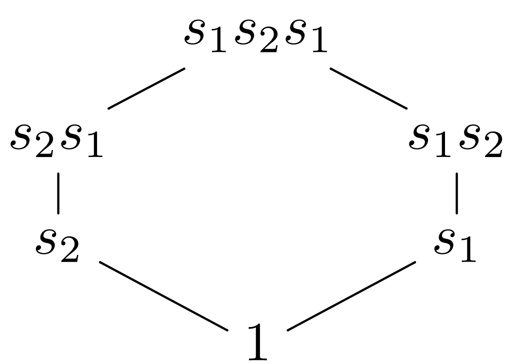
We define a positive root β to be an ***inversion of*** u if $\langle \beta, \quad \rangle$ is < 0 on uD.

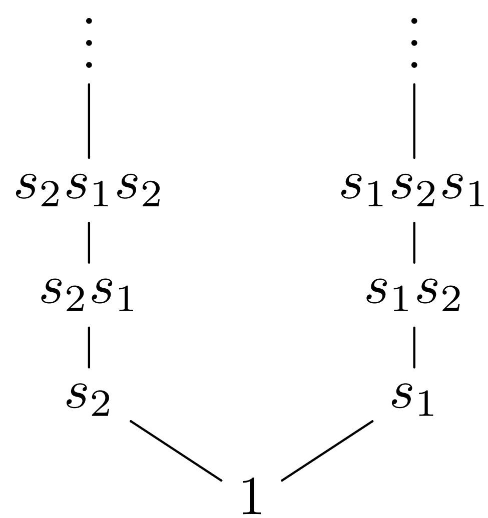
Theorem $u \leq v$ in weak order if and only if $\text{Inv}(u) \subseteq \text{Inv}(v)$.

Theorem If W is finite, then weak order is a complete lattice, meaning that every subset $\mathcal{X}$ has a unique greatest lower bound $\bigwedge \mathcal{X}$ (meet) and a unique least upper bound $\bigvee \mathcal{X}$ (join).

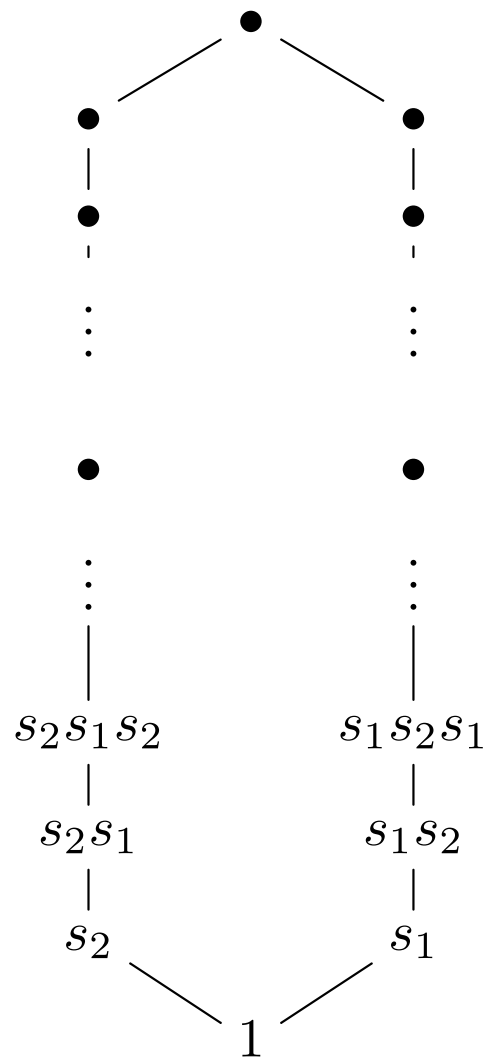
In general, weak order is a complete meet semilattice. This means:

- Every **nonempty** subset $\mathcal{X}$ of W has a meet.
- If $\mathcal{X}$ is a **bounded above** subset of W, then it has a join.





We would like to embed W into a large complete lattice.



We would like to embed W into a large complete lattice. Why?

- Complete lattices are nice!

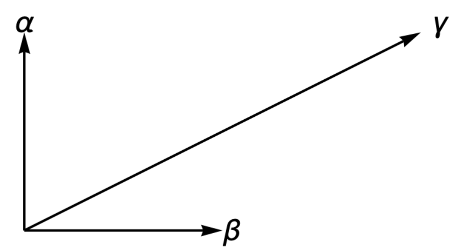
We would like to embed W into a large complete lattice. Why?

- Coxeter groups describe cluster algebras (Fomin-Zelevinsky, Reading-S., Reading-Stella, Buan-Marsh, Buan-Marsh-Reiten-Todorov, ...). When the cluster algebra has infinite type, the corresponding Coxeter group is infinite, and existing methods only describe part of the cluster complex/g-vector fan.
- Coxeter groups describe torsion classes of preprojective algebras (Ingalls-Thomas, Mizuno, Iyama-Reiten-Reading-Thomas, Demonet-IRRT, ...). When the preprojective algebra has infinite type, the corresponding Coxeter group is infinite, and existing methods only describe some of the torsion classes.
- Lam and Pylyavskyy, in their work on total positivity for loop groups, put the affine weak orders into large semilattices, which can be thought of as adding in the joins $\bigvee w_i$ for any ascending chain $w_1 < w_2 < w_3 < \cdots$.

First approach: Biclosed sets (Matthew Dyer)

Let I be a subset of Φ^+. We say that I is:

- **closed** if, for any α, β, $\gamma \in \Phi$ with $\gamma \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta$,
  $$\text{whenever } \alpha \in I \text{ and } \beta \in I \text{ then } \gamma \in I,$$
- **coclosed** if, for any α, β, $\gamma \in \Phi$ with $\gamma \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta$,
  $$\text{whenever } \alpha \notin I \text{ and } \beta \notin I \text{ then } \gamma \notin I,$$
- **biclosed** if I is closed and coclosed.



Theorem: The **finite** biclosed sets are precisely the inversion sets.

Dyer's big conjecture: The poset of biclosed sets of Φ^+, ordered by inclusion, is a complete lattice.

Let I be a subset of Φ^+. We say that I is:

- **closed** if, for any α, β, $\gamma \in \Phi$ with $\gamma \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta$,

  whenever $\alpha \in I$ and $\beta \in I$ then $\gamma \in I$,
- **coclosed** if, for any α, β, $\gamma \in \Phi$ with $\gamma \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta$,

  whenever $\alpha \notin I$ and $\beta \notin I$ then $\gamma \notin I$,

Immediate consequences of the definitions

The intersection of closed sets is closed. Every subset $X \subseteq \Phi^+$ is contained in a unique smallest closed set $\overline{X}$.

The union of coclosed sets is coclosed. Every subset $X \subseteq \Phi^+$ contains a unique largest coclosed set X°.

Dyer's big conjecture: The poset of biclosed sets of Φ^+, ordered by inclusion, is a complete lattice.

Dyer's big conjecture: The poset of biclosed sets of Φ^+, ordered by inclusion, is a complete lattice.

A better formulation (Dyer): Let $\mathcal{X}$ be any collection of biclosed subsets of Φ^+. Then $\overline{\bigcup_{I\in\mathcal{X}} I}$ is coclosed and $\left(\bigcap_{I\in\mathcal{X}} I\right)^\circ$ is closed.

If this is true, then it is immediate that $\overline{\bigcup_{I\in\mathcal{X}} I}$ is $\bigvee\mathcal{X}$ and $\left(\bigcap_{I\in\mathcal{X}} I\right)^\circ$ is $\bigwedge\mathcal{X}$.

Dyer's big conjecture: The poset of biclosed sets of Φ^+, ordered by inclusion, is a complete lattice.

A better formulation (Dyer): Let $\mathcal{X}$ be any collection of biclosed subsets of Φ^+. Then $\overline{\bigcup_{I \in \mathcal{X}} I}$ is coclosed and $\left(\bigcap_{I \in \mathcal{X}} I\right)^\circ$ is closed.

If this is true, then it is immediate that $\overline{\bigcup_{I \in \mathcal{X}} I}$ is $\bigvee \mathcal{X}$ and $\left(\bigcap_{I \in \mathcal{X}} I\right)^\circ$ is $\bigwedge \mathcal{X}$.

Note that $\bigcup_{I \in \mathcal{X}} I$ is coclosed and $\bigcap_{I \in \mathcal{X}} I$ is closed. So we can ask for even more:

Stronger conjecture (Dyer): If $Y \subset \Phi^+$ is coclosed then $\overline{Y}$ is coclosed; if $Z \subset \Phi^+$ is closed then Z° is closed.

Separability: A related but distinct concept

Let $\theta \in V^\vee$ with $\langle \beta, \theta \rangle \neq 0$ for all $\beta \in \Phi^+$. Let $X = \{\beta \in \Phi^+ : \langle \beta, \theta \rangle < 0\}$. A set X of this form is called ***separable***.

Inversion sets are separable; take $\theta \in uD$. And separable sets are biclosed. But biclosed sets don't have to be separable.

Separability: A related but distinct concept

Let $\theta \in V^\vee$ with $\langle \beta, \theta \rangle \neq 0$ for all $\beta \in \Phi^+$. Let $X = \{\beta \in \Phi^+ : \langle \beta, \theta \rangle < 0\}$. A set X of this form is called ***separable***.

Inversion sets are separable; take $\theta \in uD$. And separable sets are biclosed. But biclosed sets don't have to be separable.

There is also a more general version called ***weakly separable***. Take a basis $\theta_1, \theta_2, \ldots, \theta_r$ for $V^\vee$.

- If $\langle \beta, \theta_1 \rangle < 0$ put $\beta \in X$; if $\langle \beta, \theta_1 \rangle > 0$ put $\beta \notin X$. If $\langle \beta, \theta_1 \rangle = 0$, go the next step.
- If $\langle \beta, \theta_2 \rangle < 0$ put $\beta \in X$; if $\langle \beta, \theta_2 \rangle > 0$ put $\beta \notin X$. If $\langle \beta, \theta_2 \rangle = 0$, go the next step ...

This is more robust than separability, but doesn't make a big difference.

A key example

Look at the affine symmetric group with $n = 4$. We will compute $(s_1 s_2) \vee (s_3 s_4)$.

$\mathrm{Inv}(s_1 s_2) = \{e_2 - e_1,\ e_3 - e_1\}$ and $\mathrm{Inv}(s_3 s_4) = \{e_4 - e_3,\ e_5 - e_3\}$. What is

$$\overline{\mathrm{Inv}(s_1 s_2) \cup \mathrm{Inv}(s_3 s_4)}?$$

A key example

Look at the affine symmetric group with $n = 4$. We will compute $(s_1 s_2) \vee (s_3 s_4)$.

$\mathrm{Inv}(s_1 s_2) = \{e_2 - e_1,\ e_3 - e_1\}$ and $\mathrm{Inv}(s_3 s_4) = \{e_4 - e_3,\ e_5 - e_3\}$. What is

$$\overline{\mathrm{Inv}(s_1 s_2) \cup \mathrm{Inv}(s_3 s_4)}?$$

Some elements of the closure:

$$e_4 - e_1 = (e_4 - e_3) + (e_3 - e_1)$$

$$e_8 - e_3 = (e_4 - e_1) + (e_5 - e_3)$$

$$\cdots$$

A key example

Look at the affine symmetric group with $n = 4$. We will compute $(s_1 s_2) \vee (s_3 s_4)$.

$\mathrm{Inv}(s_1 s_2) = \{e_2 - e_1,\ e_3 - e_1\}$ and $\mathrm{Inv}(s_3 s_4) = \{e_4 - e_3,\ e_5 - e_3\}$.

$$\overline{\mathrm{Inv}(s_1 s_2) \cup \mathrm{Inv}(s_3 s_4)} =$$

$$\Big\{ e_b - e_a : a < b,\ (a, b) \equiv (1,3), (3,1), (1,2), (1,4), (3,4), (3,2) \bmod 4 \Big\}.$$

A key example

Look at the affine symmetric group with $n = 4$. We will compute $(s_1 s_2) \vee (s_3 s_4)$.

$\mathrm{Inv}(s_1 s_2) = \{e_2 - e_1,\ e_3 - e_1\}$ and $\mathrm{Inv}(s_3 s_4) = \{e_4 - e_3,\ e_5 - e_3\}$.

$$\overline{\mathrm{Inv}(s_1 s_2) \cup \mathrm{Inv}(s_3 s_4)} =$$

$$\Big\{ e_b - e_a : a < b,\ (a,b) \equiv (1,3), (3,1), (1,2), (1,4), (3,4), (3,2) \bmod 4 \Big\}.$$

This is biclosed and deserves to be $(s_1 s_2) \vee (s_3 s_4)$.

But it is not separable (or weakly separable)! Note that we have

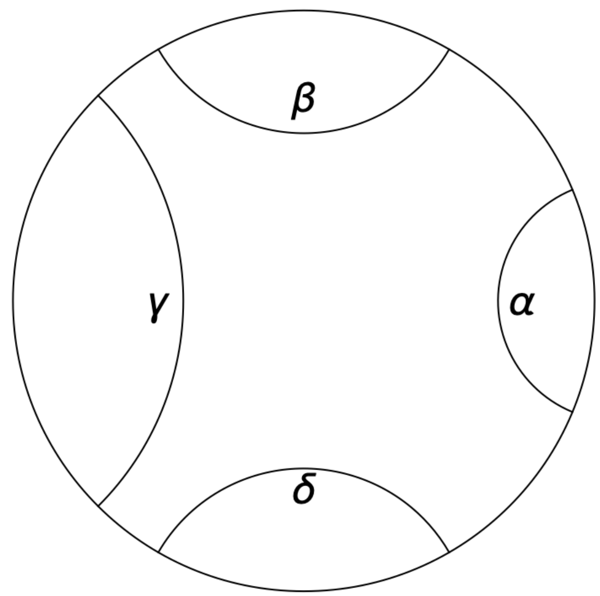
$$\overbrace{(e_5 - e_3)}^{\in X} + \overbrace{(e_3 - e_1)}^{\in X} = \delta = \overbrace{(e_6 - e_4)}^{\notin X} + \overbrace{(e_4 - e_2)}^{\notin X}.$$

Progress with Grant Barkley: We've classified biclosed sets in affine type and verified Dyer's conjecture there.

Corollary of classification: All biclosed sets in rank three affine type are weakly separable.

Open problem

Are biclosed sets in rank 3 always weakly separable? What about for the free Coxeter group?



The thing that we need to show is that, if we have four roots α, β, γ, δ as above, and I is a biclosed set, it is impossible to have α, $\gamma \in I$ and β, $\delta \notin I$. In other words, this is some sort of "noncrossing" condition.

Second approach: Shards (Nathan Reading)

Lattice congruences

Let Λ be a finite* lattice; let $\sim$ be an equivalence relation on Λ. Then $\sim$ is called a ***lattice congruence*** if $u_1 \sim u_2$ and $v_1 \sim v_2$ implies $u_1 \vee v_1 \sim u_2 \vee v_2$ and $u_1 \wedge v_1 \sim u_2 \wedge v_2$.

* Finiteness is negotiable.

Lattice congruences

Let Λ be a finite lattice; let $\sim$ be an equivalence relation on Λ. Then $\sim$ is called a ***lattice congruence*** if $u_1 \sim u_2$ and $v_1 \sim v_2$ implies $u_1 \vee v_1 \sim u_2 \vee v_2$ and $u_1 \wedge v_1 \sim u_2 \wedge v_2$.

A particularly important family of lattice congruences are the ***Cambrian congruences***. The ***Tamari congruence*** on S_n is a Cambrian congruence.

Cambrian congruences are what show up when using Coxeter groups to describe cluster algebras, and when using Coxeter groups to describe representation theory of quiver algebras.

Definition: A covering pair is a pair (u, v) of elements of Λ with $u < v$ such that there does not exist w with $u < w < v$.

In Coxeter groups, covering pairs correspond to (u, v) such that uD and vD meet along a common facet.

Theorem: A lattice congruence is determined by the list of covering pairs (u, v) for which $u \sim v$.

Definition: A covering pair is a pair (u, v) of elements of Λ with $u < v$ such that there does not exist w with $u < w < v$.

In Coxeter groups, covering pairs correspond to (u, v) such that uD and vD meet along a common facet.

Theorem: A lattice congruence is determined by the list of covering pairs (u, v) for which $u \sim v$.

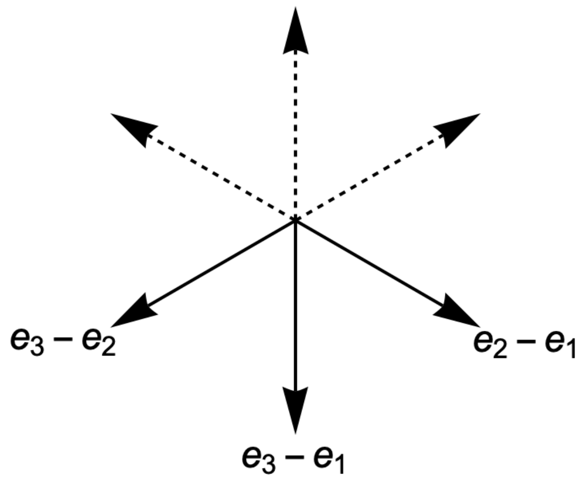
Definition: Define two covering pairs (u_1, v_1) and (u_2, v_2) to be equivalent if any congruence that collapses (u_1, v_1) also collapses (u_2, v_2) and vice versa. Let Ш be the set of covering pairs up to this equivalence.

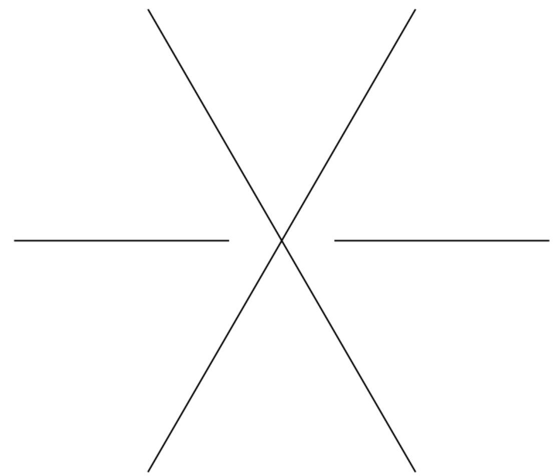
Theorem (Le Conte de Poly-Barbut) Let W be a **finite** Coxeter group. The elements of Ш are in bijection with the following sets:

1. Join irreducible elements of W. Specifically, look at the pair (j_*, j) for each join irreducible j.
2. Meet irreducible elements of W. Specifically, look at the pair (m, m_*) for each meet irreducible m.

Nathan Reading gave a third, polyhedral, way of describing Ш:

For each $\gamma \in \Phi^+$, find all cases where $\gamma \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta$ for α, $\beta \in \Phi^+$. Cut $\gamma^\perp$ along the hyperplanes $(\mathbb{R}\alpha + \mathbb{R}\beta)^\perp$. The regions of this hyperplane arrangement are called ***shards of dimension*** γ. They correspond to the elements of Ш crossing $\gamma^\perp$.





We make Reading's same definition in infinite Coxeter groups:

In infinite type, **there are more shards than there are join/meet-irreducibles.**

Theorem (S.-Thomas) There is a recursive description of shards:

- There is one shard of dimensions α_i: the whole plane $\alpha_i^\perp$.
- Suppose that $\beta = s_i(\beta')$ with $\beta \in \beta' + \mathbb{R}_{>0}\alpha_i$. Then the shard arrangement in $\beta^\perp$ is obtained by reflecting the shard arrangement in $(\beta')^\perp$ and adding in one more hyperplane, $(\mathbb{R}\beta + \mathbb{R}\alpha_i)^\perp$.

Theorem (S.-Thomas) There is a recursive description of shards:

- There is one shard of dimensions α_i: the whole plane $\alpha_i^{\perp}$.
- Suppose that $\beta = s_i(\beta')$ with $\beta \in \beta' + \mathbb{R}_{>0}\alpha_i$. Then the shard arrangement in $\beta^{\perp}$ is obtained by reflecting the shard arrangement in $(\beta')^{\perp}$ and adding in one more hyperplane, $(\mathbb{R}\beta + \mathbb{R}\alpha_i)^{\perp}$.

Theorem (S.-Thomas) There is also a representation theoretic interpretation: Shards are the stability domains of certain modules for the preprojective algebra (namely, real brick modules whose domain of stability is $(n-1)$-dimensional).

Work with Nathan Reading and Hugh Thomas

Let W be a **finite** Coxeter group. The elements of Ш are in bijection with both:

1. Join irreducible elements of W.
2. Meet irreducible elements of W.

Define partial orders $\twoheadrightarrow$ and $\hookrightarrow$ on Ш by the weak order on the join irreducibles and the meet irreducibles; this also has an interpretation in preprojective algebras. Define $x \to z$ if there exists y with $x \twoheadrightarrow y \hookrightarrow z$.

Theorem: (Reading-S.-Thomas) W is in bijection with pairs (X, Y) of subsets of Ш which are maximal with respect to the condition that there do not exist $x \in X$ and $y \in Y$ with $x \to y$. The dimensions of the shards in X and in Y are the inversions and noninversions of W respectively.

Theorem: (Reading-S.-Thomas) W is in bijection with pairs (X, Y) of subsets of Ш which are maximal with respect to the condition that there do not exist $x \in X$ and $y \in Y$ with $x \to y$. The dimensions of the shards in X and in Y are the inversions and noninversions of the element of W respectively.

Open Problem: Is there some way to impose similar relations $\twoheadrightarrow$, $\hookrightarrow$, $\to$ on Ш in infinite types such that the pairs (X, Y) to give a complete lattice.

Open Problem: In this context, If we take $\{\dim \beta : \beta \in X\}$ and $\{\dim \gamma : \gamma \in Y\}$, do we get a biclosed set and its complement?

Open Problem: If we start with a biclosed set, we can naturally associate two sets (X, Y) of shards to it. Can we say anything about the pairs we get?

Thank You!