Final Exam Study Guide
Math 156 (Calculus I), Fall 2024

1. Intuitive definition of limit and basic reasons why a limit might not exist [§2.2]
   (a) $\lim_{x\to a} f(x) = L$ means can make $f(x)$ arbitrarily close to L by making $x \neq a$ close to a
   (b) one-sided limits $\lim_{x\to a^\pm} f(x)$: they must agree for usual (two-sided) limit to exist
   (c) $\lim_{x\to a} f(x) = \pm\infty$ counts as the limit not existing
2. How to compute limits using the limit laws [§2.3, 2.5]
   (a) sum $(f+g)$, difference $(f-g)$, scaling (cf), product (fg), quotient (f/g) limit laws
   (b) how to deal with "$\frac{0}{0}$" by cancelling factors
   (c) continuous functions (pushing limit thru, and direct substitution a.k.a. "plugging in")
3. Limits at infinity and limits equal to infinity [§2.2, 2.6]
   (a) limits at $\pm\infty$ = horizontal asymptotes (typical example: $\lim_{x\to-\infty} e^x = 0$)
   (b) $\pm\infty$-valued limits = vertical asymptotes (typical example: $\lim_{x\to0^+} 1/x = \infty$)
   (c) for rational function $f(x) = P(x)/Q(x)$, $\lim_{x\to\infty} f(x)$ is 0 if degree $P(x) <$ degree $Q(x)$, is $\pm\infty$ if degree $P(x) >$ degree $Q(x)$, & is ratio of leading coefficients if degrees are =
4. The definition(s) of derivative [§2.1, 2.7, 2.8]
   (a) derivative as slope of the tangent to a curve at a point
   (b) derivative as a limit $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$
   (c) derivative as instantaneous rate of change (e.g., derivative of position is velocity)
5. Derivatives of basic functions [§3.1, 3.3, 3.6]
   (a) power functions: $d/dx(x^n) = nx^{n-1}$
   (b) exponential and logarithmic functions: $d/dx(e^x) = e^x$ and $d/dx(\ln(x)) = 1/x$
   (c) trigonometric functions: $d/dx(\sin(x)) = \cos(x)$ and $d/dx(\cos(x)) = -\sin(x)$
6. Rules for derivatives of combinations of functions [§3.1, 3.2, 3.4]
   (a) derivative is linear: $d/dx(\ a\cdot f(x) + b\cdot g(x)\ ) = a\cdot f'(x) + b\cdot g'(x)$ for $a, b \in \mathbb{R}$
   (b) product rule: $d/dx(\ f(x)\cdot g(x)\ ) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$
   (c) chain rule: $d/dx(\ f(g(x))\ ) = f'(g(x))\cdot g'(x)$
   (d) quotient rule: $d/dx(\ f(x)/g(x)\ ) = (\ g(x)\cdot f'(x) - f(x)\cdot g'(x)\ )/(g(x))^2$
   [*don't have to separately memorize quotient rule, it follows from other rules*]
7. Implicit differentiation and related rates [§3.5, 3.9]
   (a) for y defined implicitly via equation $p(x, y) = 0$, find dy/dx by taking d/dx of both sides, and use this to find the slope of the tangent at any point on the curve

(b) if two quantities $f(t), g(t)$ are related, then their rates of change df/dt, dg/dt are related: like with implicit differentiation, just differentiate the relation between $f(t)$ and $g(t)$

8. Linear approximation [§3.10]

   (a) tangent is best linear approximation to $f(x)$ near a point a: $f(x) \approx f(a) + (x-a) \cdot f'(a)$

9. Extreme values [§4.1, 4.3]

   (a) local versus absolute (global) minimum and maximum values, Extreme Value Theorem

   (b) the Closed Interval Method: extreme values of continuous f on closed interval must occur at endpoints or at critical points (values x where $f'(x) = 0$ or is not defined)

   (c) 1st and 2nd Derivative Tests for deciding if critical points are min.'s or max.'s

10. What derivatives tell us about shape of graph [§4.2, 4.3, 4.5]

    (a) $f'(x) > 0$ means f is increasing, $f'(x) < 0$ means f is decreasing

    (b) $f''(x) > 0$ means f is concave up (smile), $f''(x) < 0$ means f is concave down (frown)

11. L'Hôpital's rule [§4.4]

    (a) for indeterminate form limits (meaning "$\pm\frac{\infty}{\infty}$" or "$\frac{0}{0}$"), $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$

12. Area under the curve [§5.1, 5.2]

    (a) can approximate area under the curve $y = f(x)$ from $x = a$ to $x = b$ by the rectangle (Riemann) sum $A_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$, where $\Delta x = \frac{b-a}{n}$, $x_i = a + i \cdot \Delta x$ for $i = 0, 1, \ldots, n$, and any choice of sample points $x_i^* \in [x_{i-1}, x_i]$

    (b) usual choices: $x_i^* = x_{i-1}$ (left endpoints $A_n = L_n$); $x_i^* = x_i$ (right endpoints $A_n = R_n$); or $x^* = \frac{x_i + x_{i-1}}{2}$ (midpoints of intervals)

    (c) if $f(x)$ is continuous, all give same limit $A = \lim_{n\to\infty} A_n$, the true area under the curve

13. Definite integrals [§5.2, 5.3]

    (a) definite integral $\int_a^b f(x)\,dx$ is the area "under" the curve $y = f(x)$ from $x = a$ to $x = b$ as defined above: $A = \lim_{n\to\infty} A_n$; this counts area below the x-axis negatively

    (b) Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = F(b) - F(a) = \int f(x)\,dx\Big]_a^b$, where $F(x) = \int f(x)\,dx$ is an anti-derivative of $f(x)$

    (c) another way to think of FTC: integral of rate of change is net change (e.g., integral of velocity is displacement)

14. Anti-derivatives, a.k.a. indefinite integrals [§4.9, 5.4, 5.5]

    (a) basic anti-derivatives/indefinite integrals: $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $\int e^x\,dx = e^x + C$, $\int \frac{1}{x}\,dx = \ln(x) + C$, $\int \sin(x)\,dx = -\cos(x) + C$, $\int \cos(x)\,dx = \sin(x) + C$

    (b) integral is linear: $\int a \cdot f(x) + b \cdot g(x)\,dx = a \int f(x)\,dx + b \int g(x)\,dx$ for $a, b \in \mathbb{R}$

    (c) the u-substitution technique: can treat the "dx" in an integral as a differential, so if we let $u = g(x)$ then we can substitute $du = g'(x)\,dx$ in an integral