# Howard Math 273, HW# 2,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, November 3rd

1. Recall the Stirling numbers of the 2nd kind are $S(n,k) := \#$ set partitions of $\{1,2,\ldots,n\}$ into $k$ (non-empty) blocks. We saw $F_k(x) := \sum_{n\geq 0} S(n,k)x^n$ satisfies $F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$ for any $k \geq 1$. Find the partial fraction decomposition of $F_k(x)$, i.e., find the numbers $a_j \in \mathbb{Q}$ for which $F_k(x) = a_0 + \frac{a_1}{(1-x)} + \frac{a_2}{(1-2x)} + \cdots + \frac{a_k}{(1-kx)}$. Conclude that $S(n,k) = \sum_{j=0}^{k} a_j \cdot j^n$.
   **Hint**: Clear denominators, and then plug in $x = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}$ and finally $x = 0$.

2. *(Stanley, EC1, #2.9)* Another way to find the $a_j$ from the previous problem is using the Principle of Inclusion-Exclusion (P.I.E.), as this problem will show. Let $\widehat{S}(n,k) := k! \cdot S(n,k)$.
   (a) Explain why $\widehat{S}(n,k)$ is the number of ways to place $n$ labelled balls into $k$ labelled boxes so that all boxes are non-empty.
   (b) How many ways are there to place $n$ labelled balls into $k$ labelled boxes so that the boxes labeled $i_1, i_2, \ldots, i_j$ are empty (but the other boxes may be empty or not)?
   (c) Use parts (a), (b), and the P.I.E. to conclude that $\widehat{S}(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$.

3. *(Stanley, EC1, #2.25(a))* Let $f_i(m,n)$ be the number of $m \times n$ matrices of 0's and 1's, with a total of $i$ 1's, and with at least one 1 in each row and column. Use the P.I.E. to show that
$$\sum_{i\geq 0} f_i(m,n)t^i = \sum_{k=0}^{n} (-1)^k \binom{n}{k} ((1+t)^{n-k} - 1)^m.$$

4. *(Stanley, EC1, #2.25(b))* With $f_i(m,n)$ as in the previous problem, show that
$$\sum_{m,n\geq 0} \left( \sum_{i\geq 0} f_i(m,n)t^i \right) \frac{x^m}{m!}\frac{y^n}{n!} = e^{-x-y} \cdot \sum_{m,n\geq 0} (1+t)^{mn} \frac{x^m}{m!}\frac{y^n}{n!}.$$
   **Hint**: You can start with the formula from the previous problem, and then do some algebraic manipulations. Alternatively, you can use the theory of exponential generating functions.

5. The $q$-binomial coefficient satisfies $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \mathcal{W}_{n,k}} q^{\mathrm{inv}(w)}$, where $\mathcal{W}_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's and $k$ 1's, and $\mathrm{inv}(w)$ is the number of inversions of $w$.
   **Suppose** $n = 2m$ **is even.** Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_{q:=-1}$ (the evaluation of the $q$-binomial at $q = -1$) is equal to $\#\mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ is the subset of words $w = w_1 w_2 \ldots w_n \in \mathcal{W}_{n,k}$ that are *palindromes*, i.e., which satisfy $w_i = w_{n+1-i}$ for all $i$. Do this by defining a **sign-reversing involution**. That is, define an involution $\tau \colon \mathcal{W}_{n,k} \to \mathcal{W}_{n,k}$ satisfying:
   - $\mathrm{inv}(w)$ and $\mathrm{inv}(\tau(w))$ have opposite parity for all $w \in \mathcal{W}_{n,k}$ with $\tau(w) \neq w$;
   - $\mathrm{inv}(w)$ is even for all $w \in \mathcal{W}_{n,k}$ with $\tau(w) = w$;
   - $\#\{w \in \mathcal{W}_{n,k} \colon \tau(w) = w\} = \#\mathcal{P}_{n,k}$.