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Spring 2022, Howard Math 274:
Combinatorics II (2nd semester intro grad comb.)

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Class info:
- Meets MWF 11:10-12, online via Zoom.
- Office hrs: by appointment (email me!)
- Text: "Combinatorics: the Art of Counting" by Sagan (pdf linked to on website)
  Last semester: Ch.s 1-5, this semester: Ch.s 6-8
- Grading: There are 3 HW's (Feb., March, April)
  Beyond that, I expect you to show up to and participate in class (ask questions!)
- Disclaimer: Last semester's class was a version of a class I had taught before. This semester is new for me (may be rough around edges)

What is this class about?

We will continue our investigation/enumeration of discrete structures with a new focus on
symmetries = algebra!

The major topics (in order) will be:

(1/4 semester) group actions on combinatorial objects

(3/4 semester) symmetric functions

Let's start with group actions now!

Q: How many ways are there to color the vertices of a square w/ 3 colors (red, blue, green)?

Of course, we know A: $3^4 = 81$

But... Square has some symmetries and we might want to take these into account. We might want to consider rotations of same coloring the same:

$$\begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} = \begin{array}{ccc} B & - & R \\ | & & | \\ G & - & R \end{array} = \begin{array}{ccc} G & - & B \\ | & & | \\ R & - & R \end{array} = \begin{array}{ccc} R & - & G \\ | & & | \\ R & - & B \end{array}$$

In this case, there will certainly be fewer than 81 colorings. We might also want to consider reflections of colorings to be the same:

$$\begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} = \begin{array}{ccc} B & - & G \\ | & & | \\ R & - & R \end{array}$$

Depending on which symmetries we allow, we will get a different # of colorings.

To systematize these kinds of "counting up to symmetry" problems, we will review the algebraic notions of groups and group actions.

DEF'N A group G is a set G w/ a binary operation, multiplication $\cdot : G \times G \to G$

s.t. (1) (associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in G$,

(2) (identity) there exists an identity element $e \in G$ for which $g \cdot e = e \cdot g = g$ $\forall g \in G$

(3) (inverses) $\forall g \in G$, there exists an inverse $g^{-1} \in G$ with $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Hopefully you've seen this def'n before, but where does it come from? Important notion: <u>group action</u>!

<u>DEF'N</u> Let G be a group and X a set. An <u>action</u> of G on X (sometimes denoted $G \circlearrowright X$) is an assignment of a map $g: X \to X$ for each $g \in G$ s.t. $(gh)(x) = g(h(x))$ $\forall\ g, h \in G, x \in X$ and $e(x) = x$ $\forall x \in X$, where $e \in G$ is identity.

From now on we restrict to <u>finite</u> G and X!

<u>E.g.</u> The symmetric group S_n of <u>permutations</u> of $[n] := \{1, 2, \ldots, n\}$

$$S_n = \left\{ \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \right\}$$ "two-line notation"

has a canonical action on $[n]$.

$n=4$ $\quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot 2 = 1$

$$\Big(\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}}_{g} \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}}_{h}\Big) \cdot 2 = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}}_{gh} \cdot 2 = 4 \checkmark$$

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<u>E.g.</u> Symmetric group S_n also acts on <u>subsets</u> of $[n]$ in a natural way: for $A \subseteq [n]$, $\sigma \in S_n$

$$\sigma \cdot A := \{\sigma(a) : a \in A\}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot \{2, 3\} = \{1, 4\}$$

<u>E.g.</u> If $H \leq G$ is a <u>subgroup</u> (i.e., a subset <u>closed</u> under the multiplication) and G acts on X, then there is an <u>induced</u> action of H on X.

symmetric gp. on finite set X

Rmk: Every action $G \curvearrowright X$ is (essentially) induced from $G \le S_X$, (more precisely, from some quotient H of G $\le S_X$)

since ...

Prop: If $G \curvearrowright X$ is an action then:

(1) $g : X \to X$ is a bijection (i.e., permutation)

(2) $e : X \to X$ is the identity map.

Pf: Exercise. See Sagan.

One more kind of action will be very important for us:

E.g. If $G \curvearrowright X$ then $G \curvearrowright Y^X$ for any set Y

where $Y^X := \{$ all functions $f: X \to Y \}$

by the rule $g \cdot f(x) = f(g^{-1} x)$ $\forall x \in X, f \in Y^X$.

(The g^{-1} is to make the action satisfy $(gh)f = g(hf)$ but can be confusing at first...)

E.g. Let $X = \{1,2,3,4\}$ which we think of as vertices of square, and $Y = \{R, G, B\}$ colors. Then $f: X \to Y$, is a coloring: $f: \begin{array}{ccc} 1 & - & 2 \\ | & & | \\ 4 & - & 3 \end{array} \longrightarrow \begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array}$

Let $G = \langle (1,2,3,4) \rangle = \{ e, (1,2,3,4), (1,3)(2,4), (1,4,3,2) \}$

(subgroup generated by cycle $(1,2,3,4)$)

action Y^X:

$(1,2,3,4) \bullet \begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} = \begin{array}{ccc} B & - & R \\ | & & | \\ G & - & R \end{array}$

$\Rightarrow$ rotations of colorings of square!

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Now that we see how "counting up to symmetry" connects to group actions, we need one more definition:

<u>DEF'N</u> Let $G \curvearrowright X$ and $x \in X$. The <u>orbit</u> of x, denoted $\mathcal{O}_x$, is $\mathcal{O}_x := \{g \cdot x : g \in G\}$.

"Everything I can get to from x."

<u>Prop.</u> The orbits $\mathcal{O}_x$, $x \in X$ <u>partition</u> the set X.

<u>E.g.</u> Let $G = \langle \sigma = (1,2,3,4) \rangle \subseteq S_4$ act on $X = \{$size-2 subsets of $\{1,2,3,4\}\}$

$$\begin{tikzcd}[column sep=small] & \{1,2\} \arrow[dr, "\sigma"] & \\ \{1,4\} \arrow[ur, "\sigma"] & & \{2,3\} \arrow[dl, "\sigma"] \\ & \{3,4\} \arrow[ul, "\sigma"] & \end{tikzcd} \qquad \begin{tikzcd} \{1,3\} \arrow[r, bend left, "\sigma"] & \{2,4\} \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

there are two orbits of G acting on X

(<u>Note</u>: A group generated by a single element is called <u>cyclic</u>. <u>A</u> finite cyclic group is $\cong$ $(\mathbb{Z}/N\mathbb{Z}, +)$ for some N.)

To answer "counting up to symmetry" problems, we want to know <u>how</u> <u>many</u> <u>orbits</u> does an action $G \curvearrowright X$ have? <u>Burnside's</u> <u>Lemma</u> will give answer.

First we need one basic result in group theory.

<u>DEF'N</u> Let $G \curvearrowright X$ and $x \in X$. The <u>stabilizer</u> of x, denoted G_x is

$$G_x := \{g \in G : g \cdot x = x\}.$$

E.g. With G, X as in previous example, and $x = \{1, 3\}$, $G_x = \{e, (1,3)(2,4)\} \leq G$.

<u>Prop</u>. Any stabilizer G_x is a <u>subgroup</u> of G.

<u>Pf</u>: If $g \cdot x = x$ and $h \cdot x = x$ then $(gh) \cdot x = g(h(x)) = g(x) = x$. ☒

<u>Thm</u> (Orbit-Stabilizer Theorem)

1/26 For any $x \in X$, $\# \mathcal{O}_x \cdot \# G_x = \# G$.

<u>Pf</u>: Recall that for a subgroup $H \leq G$, a (left) <u>coset</u> of H is a set $gH := \{gh : h \in H\}$ for some $g \in G$.

Notation $G/H = \{gH : g \in G\}$ = set of cosets of H in G.

<u>Claim</u> $\exists$ bijection $\varphi : G/G_x \to \mathcal{O}_x$ given by $\varphi(g G_x) = g \cdot x$.

<u>Pf</u>: Need to check well-definedness: for $h \in G_x$, $\varphi((gh)G_x) = (gh) \cdot x = g(h(x)) = g \cdot x$. ✓

Bijectiveness: φ^{-1} given by $\varphi^{-1}(g \cdot x) = g G_x$. well-defined since if $g \cdot x = h \cdot x$, $h^{-1} g \in G_x$ so $hG_x = gG_x$ ☒

To finish proof, use another basic gp. theory result:

<u>Thm</u> (Lagrange's Thm) $\# G/H = \# G / \# H$.

<u>Pf</u>: $\exists$ bijection $\phi : H \to gH$ for any $gH \in G/H$
$\phi : h \mapsto gh$

So all gH have same size $\Rightarrow$ must be $\#G/\#H$ of them. ☒

So $\# \mathcal{O}_x = \# G/G_x = \# G / \# G_x$. ✓

E.g. Let $G = \langle (1,2,3,4) \rangle$ and $X = \{\text{size 2 subsets of } [4]\}$ as before. Then

with $x = \{1,3\}$, $\#\mathcal{O}_x \cdot \#G_x = 2 \cdot 2 = 4 = \#G$ ✓

w/ $x' = \{1,2\}$, $\#\mathcal{O}_x \cdot \#G_x = 4 \cdot 1 = 4 = \#G$ ✓

Now we are ready to give formula for # of orbits!

Lemma ("Burnside's Lemma")

The number of orbits of an action $G \curvearrowright X$ is

$$\frac{1}{\#G} \sum_{g \in G} \# X^g ,$$

where $X^g := \{x \in X : g(x) = x\}$ is the fixed-point set of $g \in G$.

Pf: Note that for any integer k,

$$\underbrace{\frac{1}{k} + \frac{1}{k} + \cdots + \frac{1}{k}}_{k \text{ times}} = 1$$

Hence for any orbit $\mathcal{O}$ of $G \curvearrowright X$ we have

$$\sum_{x \in \mathcal{O}} \frac{1}{\#\mathcal{O}_x} = \sum_{x \in \mathcal{O}} \frac{1}{\#\mathcal{O}} = 1$$

So that

$$\#\text{ of orbits of } G \curvearrowright X = \sum_{x \in X} \frac{1}{\#\mathcal{O}_x}$$

By the orbit-stabilizer Thm $\frac{1}{\#\mathcal{O}_x} = \frac{\#G_x}{\#G}$, so

$$\#\text{ of orbits} = \frac{1}{\#G} \cdot \sum_{x \in X} \# G_x .$$

Now we want to change from summing over $x \in X$ to summing over $g \in G$...

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To do that, consider the $\#G \times \#X$ matrix M whose (g,x) entry is $M_{(g,x)} = \begin{cases} 1 & \text{if } g(x)=x \\ 0 & \text{otherwise} \end{cases}$

e.g. with $G = \langle \sigma = (1,2,3,4) \rangle$, $X = \{\text{2-subsets of } [4]\}$,

$$M = \begin{array}{c} \\ e \\ \sigma \\ \sigma^2 \\ \sigma^3 \end{array} \begin{array}{c} \begin{array}{cccccc} \{1,2\} & \{2,3\} & \{3,4\} & \{1,4\} & \{1,3\} & \{2,4\} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Note that $\#G_x$ is sum of column of M corresponding to x.

So $\sum_{x \in X} \#G_x$ = sum of all columns = sum of all entries of M.

But $\#X^g$ is sum of row of M corresponding to g.

So $\sum_{g \in G} \#X^g$ = sum of all rows = sum of all entries of M = $\sum_{x \in X} \#G_x$.

Thus # of orbits $= \frac{1}{\#G} \sum_{g \in G} \#X^g$, as claimed. ☐

E.g. with $G = \langle \sigma = (1,2,3,4) \rangle$, $X = \{\text{2-subsets of } [4]\}$ as before

$$\frac{1}{\#G} \sum_{g \in G} \#X^g = \frac{1}{4}\left(\#X^e + \#X^\sigma + \#X^{\sigma^2} + \#X^{\sigma^3}\right)$$

$$= \frac{1}{4}(6 + 0 + 2 + 0) = \frac{1}{4} \cdot 8$$

$$= 2 = \#\text{ orbits of } G \curvearrowright X.$$

But what we really wanted to count was # orbits of colorings $G \curvearrowright Y^X$ induced from action $G \curvearrowright X$.

Prop. Let $G \curvearrowright X$ and consider induced action $G \curvearrowright Y^X$. Then for any $g \in G$, $f \in (Y^X)^g \Leftrightarrow f(x) = f(g^{-1}x) \ \forall x \in X$.

Pf: Recall $g \cdot f(x) = f(g^{-1}x)$, hence $gf = f$
$\Leftrightarrow f(x) = gf(x) \ \forall x \in X \Leftrightarrow f(x) = f(g^{-1}x) \ \forall x \in X$. ☒

When $G \curvearrowright X$, each $g \in G$ determines a permutation $g : X \to X$ and hence has an associated cycle structure.
Let $\boxed{c(g) := \#\text{ cycles of perm. } g : X \to X.}$

Prop. For any $g \in G$, $\#(Y^X)^g = (\#Y)^{c(g)}$.

Pf: To determine a coloring $f \in (Y^X)^g$, i.e., f w/ $f(x) = f(g^{-1}x)$ $\forall x \in X$, must choose for each cycle of g one color $y \in Y$ to give to all elts of that cycle.

Eg: w/ $g = (1,3,4)(2)(5,6)$ Choose
- color for 1, 3, 4
- color for 2
- color for 5, 6.

Total # of choices $= \underbrace{\#Y \cdot \#Y \cdots \#Y}_{\text{each cycle}} = \#Y^{c(g)}$. ☒

Cor ("Unweighted Pólya counting"):
Let $G \curvearrowright X$ and consider induced coloring action $G \curvearrowright Y^X$.
Then # orbits of $G \curvearrowright Y^X = \dfrac{1}{\#G} \displaystyle\sum_{g \in G} (\#Y)^{c(g)}$.

E.g. We can now answer our motivating Q:
How many colorings of vertices of square, with 3 colors, up to rotation?

Take $G = \langle \sigma = (1,2,3,4) \rangle \circlearrowright X = [4]$ w/ colors set $Y = \{R, G, B\}$. Then # orbits $G \circlearrowright Y^X$

$$= \frac{1}{\# G} \sum_{g \in G} (\# Y)^{c(g)} = \frac{1}{4}\left((\# Y)^{c(e)} + (\# Y)^{c(\sigma)} + (\# Y)^{c(\sigma^2)} + (\# Y)^{c(\sigma^3)} \right)$$

$$= \frac{1}{4}\left(3^4 + 3^1 + 3^2 + 3^1 \right) = \frac{1}{4}(96) = \underline{24}.$$

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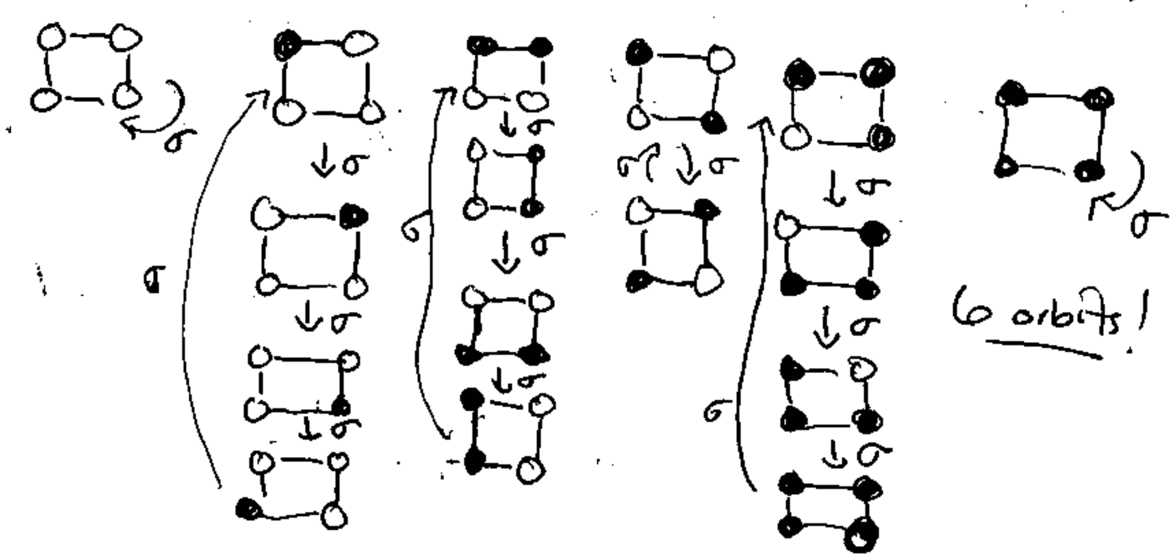
Let's take same G, X as last example, w/ $k = 2$.

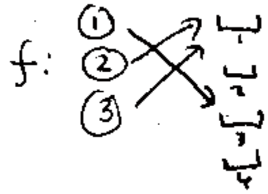
Then #colorings up to symmetry $= \frac{1}{4}\left(2^4 + 2^1 + 2^2 + 2^1 \right)$
$= \frac{1}{4}(24) = 6.$

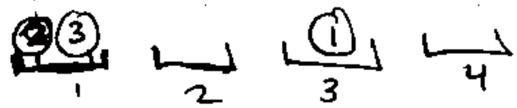
This is small enough that we can check:



E.g. for a different kind of example, let's take $G = S_n$ full symmetric gp. acting on $X = [n]$ (in natural way), and $Y = [k]$. In this case,

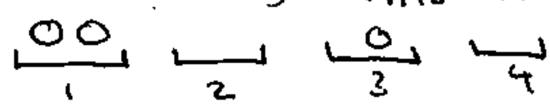
$$Y^X = \{ \text{functions } f: [n] \to [k] \}$$
$$= \{ \text{ways of putting } n \text{ labeled balls into } k \text{ labeled bins} \}.$$

e.g. 

= 

and $\{\text{orbits of } G \circlearrowright Y^X\}$ w/ think: "modding out by S_n action on ball labels"

$$= \{ \text{ways of putting } n \text{ unlabeled balls into } k \text{ labeled bins} \}$$

e.g. $G \cdot f$ = 

Last semester we saw using "stars and bars" that $\#$ orbits of $G \circlearrowright Y^X = \binom{n+k-1}{n}$.

We can also see this formula from the unweighted Pólya counting, which says

$$\#\text{orbits } G \circlearrowright Y^X = \frac{1}{\#G} \sum_{g \in G} (\#Y)^{c(g)} = \frac{1}{n!} \sum_{\sigma \in S_n} k^{c(\sigma)}$$
$$= \frac{1}{n!} \sum_{j=1}^{n} c(n,j) \cdot k^j, \quad \text{where}$$

$c(n,j) = \#\{\text{perms } \sigma \text{ in } S_n \text{ w/ } j \text{ total cycles}\}$
$=$ (unsigned) Stirling #'s of 1st kind.

Last semester $\Rightarrow \sum_{j=1}^{n} c(n,j) t^j = t(t+1)\cdots(t+(n-1))$.

$$\Rightarrow \#\text{ orbits} = \frac{1}{n!} \cdot k(k+1)\cdots(k+(n-1)) = \binom{n+k-1}{n} //$$