Weighted Pólya counting, a.k.a. Pólya-Redfield enumeration

Notice that the example of

$G = \langle \sigma = (1,2,3,4) \rangle \curvearrowright$ colorings of $[4]$ w/ 2 colors (e.g. black+white)

is the same as

$G \curvearrowright$ subsets of $[4]$

(Just use subset of black vertices.)
But we also know that an action on subsets like this preserves size of subset, e.g. we looked at

$G \curvearrowright$ size 2 subsets of $[4]$

which in the language of colorings would be same as

$G \curvearrowright$ colorings of $[4]$ w/ 2 vertices white + 2 vertices black.

In general we might want to keep track of precise number of each color used, as in:

Q: Up to rotation, how many colorings of vertices of square have exactly 2 red, 1 blue, 1 green vertex?

2/1 To answer this question, we need more notation.
For a coloring $f : X \to Y = \{1, 2, \ldots k\}$ define monomial $\vec{y}^{\,f} := \prod_{x \in X} y_{f(x)} \in \mathbb{C}[y_1, y_2, \ldots, y_k]$

e.g. coloring $f = \begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} \rightsquigarrow y_1^2 y_2 y_3$ if we decide $\begin{pmatrix} R = 1 \\ B = 2 \\ G = 3 \end{pmatrix}$

Notice: If $G \curvearrowright X$ then $\vec{y}_f = \vec{y}_{g \cdot f} \quad \forall g \in G$.

DEF'N Let $G \curvearrowright X$ and hence on colorings Y^X.
The pattern inventory polynomial of $G \curvearrowright Y^X$ is

$$P(y_1, y_2, \ldots, y_k) = \sum_{\mathcal{O}} \vec{y}^{\mathcal{O}} \in \mathbb{C}[y_1, \ldots, y_k]$$

where the sum is over all orbits $\mathcal{O}$ of $G \curvearrowright Y^X$
and $\vec{y}^{\mathcal{O}} := \vec{y}^f$ for any coloring $f \in \mathcal{O}$.

e.g. For $G = \langle \sigma = (1,2,3,4) \rangle \curvearrowright X = [4]$ and $Y = \{\circ, \bullet\} \text{``=''} \{1,2\}$,

$$P = 1y_1^4 + 1y_1^3 y_2 + 2y_1^2 y_2^2 + 1y_1 y_2^3 + 1y_2^4$$

Given the pattern inventory poly. P, we can then answer questions like: "how many (symmetry classes of) colorings use 2 white + 2 black vertices?" by extracting coefficients.

So our goal will now be to give a formula for $P(y_1, \ldots, y_k)$.
To do that, we need to keep track of more refined cycle information of elements $g: X \to X$, $g \in G$.
Set $c_i(g) := \#$ i-cycles of permutation $g: X \to X$.
("=" cycles of size i)

DEF'N The cycle index polynomial of $G \curvearrowright X \text{``=''} [n]$ is

$$Z_G(t_1, t_2, \ldots, t_n) = \frac{1}{\#G} \sum_{g \in G} \underbrace{\prod_{i=1}^{n} t_i^{c_i(g)}}_{\vec{t}^{\,g}} \in \mathbb{C}[t_1, \ldots, t_n]$$

This is the key to Pólya counting!

e.g. With $G = \langle \sigma = (1,2,3,4) \rangle \curvearrowright X = [4]$, have

$$Z_G = \frac{1}{4}\Big(\underbrace{t_1^{\,4}}_{e=(1)(2)(3)(4)} + \underbrace{2t_4}_{\substack{\sigma=(1,2,3,4)\\ \sigma^3=(4,3,2,1)}} + \underbrace{t_2^{\,2}}_{\sigma^2=(1,3)(2,4)} \Big)$$

Thm (Pólya-Redfield enumeration theorem)
The pattern inventory polynomial of $G \curvearrowright Y^X$ is

$$P = Z_G\Big(\underbrace{\sum_{i\in Y} y_i}_{t_1},\ \underbrace{\sum_{i\in Y} y_i^2}_{t_2},\ \underbrace{\sum_{i\in Y} y_i^3}_{t_3}, \ldots, \underbrace{\sum_{i\in Y} y_i^n}_{t_n} \Big).$$

e.g. Let $G = \langle \sigma = (1,2,3,4) \rangle \curvearrowright X = [4]$ and consider set of colors $Y =$ "$\{R, G, B\}$" $= \{1,2,3\}$.

Thm $P = \frac{1}{4}\left((y_1+y_2+y_3)^4 + 2(y_1^4+y_2^4+y_3^4) + (y_1^2+y_2^2+y_3^2)^2 \right)$

$$= \cdots = y_1^4 + y_2^4 + y_3^4 + y_1^3y_2 + y_1^3y_3 + y_2^3y_1 + y_2^3y_3$$
$$+ y_3^3y_1 + y_3^3y_2 + 2y_1^2y_2^2 + 2y_1^2y_3^2$$
$$+ 2y_2^2y_3^2 + 3y_1^2y_2y_3 + 3y_2^2y_1y_3 + 3y_3^2y_1y_2.$$

lots of algebra!

To figure out how many colorings have 2 R, 1 B, 1 G, we extract coeff. of $y_1^2 y_2 y_3$ from P;

A: $[y_1^2 y_2 y_3]P = 3$ colorings w/ 2R, 1B, 1G.

Note: Setting $y_i := 1$ for all $i \in Y$, we recover the unweighted Pólya counting formula for total number of colorings (ignoring patterns). //

2/4

Pf of Pólya-Redfield thm:

The proof is very similar to unweighted result; we just need to make sure we keep track of weights.

- First observe that for any orbit $\mathcal{O}$ of $G \curvearrowright Y^X$,

$$\vec{y}^{\mathcal{O}} = \sum_{f \in \mathcal{O}} \vec{y}^f / \#\mathcal{O}_f \quad , \quad \text{so}$$

$$P = \sum_{\mathcal{O}} \vec{y}^{\mathcal{O}} = \sum_{f \in Y^X} \frac{\vec{y}^f}{\#\mathcal{O}_f} = \frac{1}{\#G} \sum_{f \in Y^X} \# G_f \cdot \vec{y}^f,$$

where as before we used the Orbit-Stabilizer Thm.

By the same "summing over rows" vs. "summing over columns" trick, applied to matrix

$$M(g,f) = \begin{cases} \vec{y}^f & \text{if } g \cdot f = f \\ 0 & \text{otherwise} \end{cases}, \quad \text{get that}$$

$$P = \sum_{\mathcal{O}} \vec{y}^{\mathcal{O}} = \frac{1}{\#G} \sum_{g \in G} \sum_{f \in (Y^X)^g} \vec{y}^f .$$

- So we again need to think about $(Y^X)^g$ for $g \in G$.

Recall that $f \in (Y^X)^g \Leftrightarrow f(x) = f(x')$ whenever x, x' belong to same cycle of $g: X \to X$

e.g. $g = \underbrace{(x_1, \overset{t_3}{x_2}, x_3)}_{\text{color all red or all blue or all green}} \underbrace{(\overset{t_1}{x_4})}_{\text{color red or blue or green}} \underbrace{(x_5, \overset{t_2}{x_6})}_{\text{color both red or both blue or both green}} : X \to X$

$\Downarrow$

$$\sum_{f \in (Y^X)^g} \vec{y}^f = (y_1^3 + y_2^3 + y_3^3) \cdot (y_1 + y_2 + y_3) \cdot (y_1^2 + y_2^2 + y_3^2)$$

So in general $\displaystyle\sum_{f \in (Y^X)^g} \vec{y}^f = \prod_{\substack{\text{cycles} \\ c \text{ of } g: X \to X}} \sum_{y \in Y} y_i^{\text{size of } c}$.

This precisely means $P(y_1, \ldots, y_K) = Z_G(\sum_i y_i, \sum_i y_i^2, \ldots, \sum_i y_i^n)$. ∎

Cor Let $G \curvearrowright \overset{[n]}{\overset{\shortparallel}{X}}$. Then

$$\sum_{k=0}^{n} \#\left(\begin{matrix}\text{orbits of}\\ G \curvearrowright \text{size } k \text{ subsets of } X\end{matrix}\right) \cdot t^k = Z_G(1+t, 1+t^2, \ldots, 1+t^n).$$

Pf: Use 2 colors in weighted Pólya counting. ▨

2/9 e.g. Recall that a (simple) graph consists of a vertex set V and a set of edges E, unordered pairs of vertices.

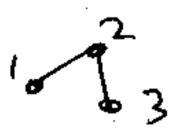
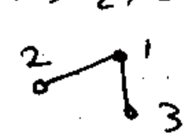
$G =$  $\Rightarrow V = [5],\ E = \{\{1,2\},\{2,3\},\{4,5\}\}$.

Q: How many graphs G with vertex set $V = [n]$?

A: $2^{\binom{n}{2}}$ since there are $\binom{n}{2}$ possible edges, and we can choose any subset of edges.

But... what if we want to count unlabeled graphs, i.e., graphs up to isomorphism?

DEF'N An isomorphism between graphs $G = (V, E)$ and $G' = (V', E')$ is a bijection $\phi: V \to V'$ on vertices s.t. $\{i,j\} \in E \Leftrightarrow \{\phi(i), \phi(j)\} \in E'$.

e.g.  $= G \sim$  $= G'$.

Q: How many isomorphism classes of graphs w/ n vertices are there?

Even better, what is $\displaystyle\sum_{G,\ \text{graph on } n \text{ vertices}} t^{\#\text{edges}(G)}$?

$\underline{A}$: By weighted Pólya counting, answer is

$$Z_G(1+t, 1+t^2, \ldots, \cancel{}, 1+t^{\binom{n}{2}})$$

where $G = S_n \circlearrowright X = \{\text{size 2 subsets of } [n]\}$.

$\boxed{\text{WARNING! Not } X = [n]}$

Let's first consider case $n=3$:

cycle type λ	$\sigma \in S_n$ w/ this type	cycle structure of $\sigma: X \to X$	monomial $\vec{t}^{\sigma}$	# $\sigma \in S_n$ w/ type λ
$(1,1,1)$	$e=(1)(2)(3)$	$(\{1,2\})(\{1,3\})(\{2,3\})$	t_1^3	1
$(2,1)$	$(1,2)(3)$	$(\{1,3\},\{2,3\})(\{1,2\})$	$t_2 t_1$	3
(3)	$(1,2,3)$	$(\{1,2\},\{1,3\},\{2,3\})$	t_3	2

So $Z_G(t_1,t_2,t_3) = \frac{1}{3!}(t_1^3 + 3t_2t_1 + 2t_3)$

and $Z_G(1+t, 1+t^2, 1+t^3) = \frac{1}{6}((1+t)^3 + 3(1+t^2)(1+t) + 2(1+t^3))$

$= t^3 + t^2 + t + 1$.

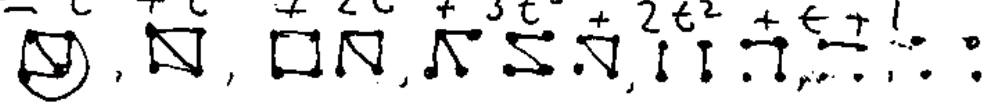
$\leftarrow$ g.f. of graphs on $n=3$ vertices, by #edges.

$n=4$: λ	$\sigma \in S_n$	cyclestructure of $\sigma: X \to X$	$\vec{t}^{\sigma}$	# $\sigma \in S_n$
$(1,1,1,1)$	$e=(1)(2)(3)(4)$	$(12)(13)(14)(23)(24)(34)$	t_1^6	1
$(2,1,1)$	$(1,2)(3)(4)$	$(12)(13,23)(14,24)(34)$	$t_2^2 t_1^2$	$\binom{4}{2}=6$
$(2,2)$	$(1,2)(3,4)$	$(12)(13,24)(14,23)(34)$	$t_2^2 t_1^2$	$\binom{4}{2}/2 = 3$
$(3,1)$	$(1,2,3)(4)$	$(12,23,13)(14,24,34)$	t_3^2	$4 \cdot 2 = 8$
(4)	$(1,2,3,4)$	$(12,23,34,14)(13,24)$	$t_4 t_2$	$3! = 6$

So $Z_G(t_1,t_2,t_3,t_4) = \frac{1}{4!}(t_1^6 + 9t_2^2t_1^2 + 8t_3^2 + 6t_4t_2)$

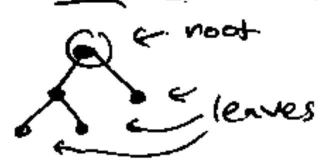
and $Z_G(1+t, 1+t^2, 1+t^3) = \frac{1}{24}((1+t)^6 + 9(1+t^2)^2(1+t)^2$
$+ 8(1+t^3)^2 + 6(1+t^4)(1+t^2))$

$= t^6 + t^5 + 2t^4 + 3t^3 + 2t^2 + t + 1$



$\swarrow$ g.f. for graphs on 4 vertices

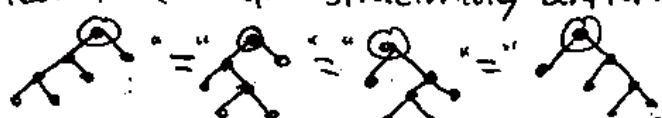
Cultural aside on trees: Pólya developed Pólya counting to enumerate trees, motivated by problems in molecular chemistry!

A rooted binary tree looks like: 

each node has 2 or 0 children

rooted binary trees w/ n+1 leaves = Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ ← last semester!

e.g. $C_3 = 5$ ⇝ 

But... what is we wanted to count "structurally different" binary trees, i.e. 

Let $a_n :=$ # structurally different rooted binary trees w/ n+1 leaves

$n = 0, 1, 2, 3, 4, 5, \ldots$
$C_n = 1, 1, 2, 5, 14, 42, \ldots$
$a_n = 1, 1, 1, 2, 3, 6, \ldots$

Set $C(x) := \sum_{n\geq 0} C_n x^n$ and $A(x) := \sum_{n\geq 0} a_n x^n$ $\mathbb{Z}_2^n$ (A(x), A(x^2)) ''S_n'' life

We saw $C(x) = 1 + x \cdot C(x)^2$

Pólya counting ⟹ $A(x) = 1 + \frac{x}{2}\left(A(x)^2 + A(x^2)\right)$ ← account for symmetry!

Even leads to asymptotics for all trees!:

Thm (Otter, 1948) Let $t_n :=$ # unlabeled, unrooted trees on n vertices

Then $t_n \sim C\alpha^n n^{-5/2}$ w/ $\alpha \approx 2.955\ldots$, $C \approx 0.5349\ldots$

Compare: $n^{n-2}/n! \sim \frac{1}{\sqrt{2\pi}} e^n n^{-5/2}$

Cayley's formula for labeled trees! ($e \approx 2.71\ldots$)