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<u>New</u> <u>topic</u>: <u>Symmetric functions</u>!

Today we will start our investigation of symmetric functions, which will occupy us for the rest of the semester. This is a <u>huge</u> topic, which could more than fill a semester

Q: Why care about symmetric functions?

<u>Real answer</u>: the combinatorics of s.f.'s controls...

- "the representation theory of the <u>symmetric group</u> S_n"
- "the repr'n thry of the <u>general linear group</u> $GL_n(\mathbb{C})$"
- "the cohomology of the <u>Grassmannian</u> $Gr_{k,n}(\mathbb{C})$"

(will not discuss any of this, until maybe the very end of the semester. For now these will just be buzz words.

<u>Q</u>: What <u>are</u> symmetric functions?

<u>A</u>: Let's start by describing <u>symmetric</u> <u>polynomials</u>.

Recall $\mathbb{C}[x_1, x_2, \ldots, x_n] = \{$polynomials in n variables $x_1, x_2, \ldots, x_n$ (w/ $\mathbb{C}$-coeff's$\}$.

The symmetric group S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$ by <u>permuting</u> <u>indices</u> of the variables:

$$\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

<u>e.g.</u> $n=3$ $\quad (1,3,2)\cdot(x_1^2 x_2 + 2x_3) = x_3^2 x_1 + 2x_2$.

<u>DEF'N</u> A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called <u>symmetric</u> if $\sigma \cdot f = f$ for all $\sigma \in S_n$, i.e., if f is <u>invariant</u> under the whole action of the symmetric group.

Sometimes use $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$ to denote set of symmetric poly's.

e.g. $n=3$, $f = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + 2x_1 + 2x_2 + 2x_3 \in \mathbb{C}[x_1,x_2,x_3]^{S_3}$.

How do symmetric polynomials arise "in nature"?
Here are two instances:

1) Let $f \in \mathbb{C}[x]$ be a monic, univariate polynomial (in variable x)
So $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
for coefficients $a_{n-1},\ldots,a_1,a_0 \in \mathbb{C}$.
By fund. thm. of algebra we know f has n roots (w/ multiplicity)
so that $f(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$,
where $r_1, r_2, \ldots, r_n \in \mathbb{C}$ are the roots (w/ mult.).

Q: How do we express coeff's a_i in terms of the roots r_j?

e.g. $n=3$ $f = (x-r_1)(x-r_2)(x-r_3) =$
$$x^3 - (r_1+r_2+r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - (r_1r_2r_3).$$

i.e., $a_k = \left(\sum_{1\le i_1<i_2<\cdots<i_{n-k}\le n} r_{i_1}\cdot r_{i_2}\cdots r_{i_{n-k}}\right)\cdot(-1)^{n-k}$

In other words, the coefficients are symmetric polynomials in the roots

(in fact, these are important examples of sym. poly's called "elementary symmetric polynomials.")

Think: Characteristic polynomial $\det(\lambda I - M)$ of square matrix M, and its eigenvalues.

2) Think about Pólya counting: $G \curvearrowright X$ and on Y^X. ($Y = \{1,2,\dots,k\}$)

We considered the pattern inventory polynomial

$$P(y_1,\dots,y_k) = \sum_{\text{orbits } \Theta \text{ of } G \curvearrowright Y^X} \vec{y}^{\,\Theta} \in \mathbb{C}[y_1,\dots,y_k]$$

This is a symmetric polynomial in the $y_1,\dots,y_k$.

Why? A1: Given any (G-equiv. class) of coloring, can always "relabel" colors to produce another one!

$$\begin{matrix} R - R \\ | \quad\;\; | \\ B - G \end{matrix} \quad \xrightarrow[\substack{R \mapsto B \\ G \mapsto R \\ B \mapsto G}]{} \quad \begin{matrix} B - B \\ | \quad\;\; | \\ G - R \end{matrix}$$

A2: By the main thm of Pólya theory.

$$P(y_1,\dots,y_k) = Z_G\Big(\sum_i y_i,\ \sum_i y_i^2,\ \dots,\ \sum_i y_i^m\Big)$$

and the things we're plugging in are all clearly symmetric polynomials (they are called "power sum symmetric polynomials.")

2/14 Okay, so now we have a feel for symmetric polynomials. But... what are symmetric functions?

Basically, we want to study $\mathbb{C}[x_1,\dots,x_n]^{S_n}$ "for all values of n at once," or another way to think of it is that we want to look at

$$\text{“} \lim_{n\to\infty} \mathbb{C}[x_1,\dots,x_n]^{S_n} \text{”}.$$

for the "functions" bit in "sym. functions" you should think of "generating functions," i.e., power series (they will not really be functions).

We let $\mathbb{C}[[x_1, x_2, \ldots]] := \left\{ \begin{array}{l} \text{ring of formal power series in (w/ } \mathbb{C}\text{-coeff's)} \\ \text{infinitely many variables } x_1, x_2, x_3, \ldots \end{array} \right\}$

An element $f \in \mathbb{C}[[x_1, x_2, \ldots]]$ looks like

$$f = \sum \alpha_{i_1, i_2, \ldots, i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} \quad (\text{for } \alpha \in \mathbb{C})$$

We want to limit somewhat the kind of power series that we look at. Recall that the degree of a monomial $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$ is $i_1 + \cdots + i_k$.

Say $f \in \mathbb{C}[[x_1, x_2, \ldots]]$ is homogeneous of degree n if it's a (possibly infinite) linear combination of monomials of degree n,

e.g. $f = \sum_{i_1 < i_2} x_{i_1}^2 x_{i_2}$ is homo. of deg. $= 3$.

Say $f \in \mathbb{C}[[x_1, x_2, \ldots]]$ has bounded degree if it's a finite linear combination of homogeneous power series.

e.g. $f = \sum_{i_1} x_i^5 + \sum_{i_1 < i_2} x_{i_1}^2 x_{i_2}$ is bounded degree.

Every polynomial $\sigma \in S_n$ in every symmetric gp. S_n acts on (bounded degree elts. of) $\mathbb{C}[[x_1, x_2, \ldots]]$ in the natural way by permuting indices.

DEF'N The ring of symmetric functions is $\underset{\text{Stanley's notation}}{\Lambda} = \underset{\text{Sagan's notation}}{Sym} := \left\{ \begin{array}{l} f \in \mathbb{C}[[x_1, x_2, \ldots]] \\ \text{of bounded degree} \end{array} : \sigma \cdot f = f \ \forall \sigma \in S_n \ \forall n \right\}$

We have "$\mathrm{Sym} = \lim_{n\to\infty} \mathbb{C}[x_1,\dots,x_n]^{S_n}$" in sense that:

<u>Prop</u>. for any $f(x_1,x_2,\dots)\in \mathrm{Sym}$, $f(x_1,x_2,\dots,x_n,0,0,0,\dots) \in \mathbb{C}[x_1,\dots,x_n]^{S_n}$

(Set all $x_i := 0$ for $i > n$)

<u>Pf</u>: Exercise. Think about how bounded degree condition forces $f(x_1,\dots,x_n,0,0,\dots)$ to be a <u>polynomial</u>. ∎

What do elements of Sym look like?

<u>e.g.</u> $f = \sum_i x_i^3 + \sum_{i,j} x_i^4 x_j^3 + \sum_{i<j} x_i x_j \in \mathrm{Sym}$.

Notice that every $f \in \mathrm{Sym}$ is a finite linear combination of homogeneous elts of Sym, i.e.,

$$\mathrm{Sym} = \bigoplus_{n\geq 0} \mathrm{Sym}(n) \quad \text{(Vector space direct sum)}$$

where $\mathrm{Sym}(n) = \{f\in \mathrm{Sym} ; f \text{ is homo. of deg.} = n\}$.

Sym is an infinite dimensional $\mathbb{C}$-v.s., but each graded component is fin. dim'l, and now we will describe a basis. ← (actually several bases)

2/16 Recall that an <u>integer</u> <u>partition</u> $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0)$ is a weakly decreasing seq. of integers, and we say λ is a partition of $n = |\lambda|$ (or $\lambda \vdash n$) if

$n = \lambda_1 + \lambda_2 + \dots + \lambda_k$, e.g. $(4,3,3,1)$ is a partition of 11

Partitions are fundamental in combinatorics of Sym, since $\dim_{\mathbb{C}} \mathrm{Sym}(n) = p(n) = \#$ partitions $\lambda \vdash n$.

<u>DEF'N</u> Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. The <u>monomial symmetric function</u> m_λ is

$$m_\lambda(x_1, x_2, \dots) = \sum_{i_1, i_2, \dots, i_k \,:\, i_j < i_{j+1} \text{ if } \lambda_j = \lambda_{j+1}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},$$

where the sum includes each monomial w/ exponent sequence $(\lambda_1, \dots, \lambda_k)$ exactly once.

<u>e.g.</u> $m_{(2,2,1)} = x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_4 + \cdots + x_2^2 x_3^2 x_1 + \cdots$

It's easy to see that $m_\lambda \in \mathrm{Sym}$. In fact...

<u>Thm</u> The monomial symmetric functions m_λ for $\lambda \vdash n$ form a <u>basis</u> of $\mathrm{Sym}(n)$.

<u>Pf:</u> As mentioned, it is clear from the definition that m_λ for $\lambda \vdash n$ is a sym. function of deg. $= n$. That the m_λ are linearly independent is also easy to see since their <u>supports</u> are disjoint. Here the <u>support</u> of a f.p.s. $f \in \mathbb{C}[[x_1, x_2, \dots]]$ is the set of monomials that appear in with nonzero coefficient. To show the m_λ span $\mathrm{Sym}(n)$: choose any $f \overset{\neq 0}{\in} \mathrm{Sym}(n)$. Since $f \neq 0$, there is some monomial (of deg. $= n$) in its support; by permuting the indices we must have in$f$ a monomial of form $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ w/ $\lambda_1 \geq \cdots \geq \lambda_k$. Let $\alpha \neq 0$ be the coeff. of $\underline{\nearrow}$ in f. Then

$f - \alpha m_\lambda$ is still a sym. fun. of deg. $= n$, and it has strictly fewer mono.'s of form $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$, $\mu_1 \geq \cdots \geq \mu_\ell$ in its support. By induction, $f \in \mathrm{Span}_{\mathbb{C}}\{m_\lambda : \lambda \vdash n\}$. ▨

Other important bases of Sym

The ring of sym. fun's has several important bases, and understanding the relationship between the various bases is a main topic in sym. fun. theory.

DEF'N The k^{th} elementary symmetric function is

$$e_k(x_1,x_2,\dots) := \sum_{i_1<i_2<\dots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \quad (= m_{(1,1,\dots,1)})$$

The k^{th} complete homogeneous symmetric function is

$$h_k(x_1,x_2,\dots) := \sum_{i_1\le i_2\le\dots\le i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \quad \left(= \sum_{\lambda\vdash k} m_\lambda\right)$$

The k^{th} power sum symmetric function is

$$p_k(x_1,x_2,\dots) := \sum_i x_i^k \quad (= m_{(k)}).$$

2/18 For e_k and h_k, also have nice gen. fun. representations:

Prop. a) $\sum_{k\ge0} e_k(x_1,x_2,\dots)t^k = \prod_{i\ge1}(1+x_it)$

b) $\sum_{k\ge0} h_k(x_1,x_2,\dots)t^k = \prod_{i\ge1}\frac{1}{1-x_it}$.

Pf: When we expand $\prod_{i\ge1}(1+x_it)$ we get all monomials made up of distinct variables in x_i, multiplied by t^a where a = deg. of monomial.

Similarly, when we expand $\prod_{i\ge1}\frac{1}{(1-x_it)} = \prod_{i\ge1}(1+x_it+x_i^2t^2+\dots)$ we get all monomials in the variables x_i, multiplied by $t^{\text{deg. of monomial}}$. $\boxtimes$

But the e_k, h_k, or p_k cannot be a basis of Sym, because that's just one sym. fun. for each degree.

To get bases from the e_k, h_k, and p_k, need to multiply!

DEFN Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ be a partition. Define the corresponding elementary, complete homo., and power sum sym fun's to be

$$e_\lambda(x_1, x_2, \dots) = e_{\lambda_1} \cdot e_{\lambda_2} \cdot \cdots \cdot e_{\lambda_k},$$
$$h_\lambda(x_1, x_2, \dots) = h_{\lambda_1} \cdot \cdots \cdot h_{\lambda_k},$$
$$p_\lambda(x_1, x_2, \dots) = p_{\lambda_1} \cdot \cdots \cdot p_{\lambda_k}.$$

e.g. Say $\lambda = (2,1) \vdash 3$. Then

$$e_{(2,1)} = e_2 \cdot e_1 = (x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$
$$= (x_1^2 x_2 + 3x_1x_2x_3 + \cdots) = m_{(2,1)} + 3m_{(1,1,1)}$$
$$h_{(2,1)} = h_2 \cdot h_1 = (x_1x_2 + x_1x_3 + x_2x_3 + \cdots + x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$
$$= (2x_1^2x_2 + 3x_1x_2x_3 + \cdots + x_1^3 + \cdots) = m_{(3)} + 2m_{(2,1)} + 3m_{(1,1,1)}$$
$$p_{(2,1)} = p_2 \cdot p_1 = (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots)$$
$$= (x_1^3 + \cdots + x_1^2x_2 + \cdots) = m_{(3)} + m_{(2,1)}$$

Thm For each $n \geq 1$, the sets

$$\{e_\lambda : \lambda \vdash n\}, \quad \{h_\lambda : \lambda \vdash n\}, \quad \{p_\lambda : \lambda \vdash n\}$$

are each bases of $Sym(n)$.

Rmk: Can also rephrase this thm as saying

$$Sym \cong \mathbb{C}[e_1, e_2, \dots] \cong \mathbb{C}[h_1, h_2, \dots] \cong \mathbb{C}[p_1, p_2, \dots]$$

is a polynomial ring in the e_k, h_k, or p_k.

For the e_k, this is called the "fundamental Thm. of Sym. Fun's" and was proved by Newton.

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Pf: Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ be two partitions of n.
We say $\lambda > \mu$ in lexicographic order if there is some j such that $\lambda_i = \mu_i \ \forall\, i < j$ and $\lambda_j > \mu_j$.

e.g. $(3,2,2,1) > (3,2,1,1,1)$.

To show $\{p_\lambda : \lambda \vdash n\}$ is a basis of $Sym(n)$, consider writing p_λ as a lin. comb. of m_μ.

Claim: $p_\lambda = \underset{\neq 0}{\alpha^\lambda_\lambda} m_\lambda + \sum_{\substack{\mu > \lambda \\ \text{(in lex. order)}}} \alpha^\lambda_\mu m_\mu$ for coeff's $\alpha^\lambda_\mu \in \mathbb{C}$

In other words, the smallest m_μ (in lex. order) appearing in p_λ is m_λ. Why? Consider expanding

$$p_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_1} + \dots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \dots) \cdots (x_1^{\lambda_k} + x_2^{\lambda_k} + \dots)$$

To find smallest m_μ that appears in p_λ, we want to find smallest monomial of form $x_1^{\mu_1} x_2^{\mu_2} \cdots x_\ell^{\mu_\ell}$ in support.
But the way to make lex. smallest monomial is to choose $x_1^{\lambda_1}$ from 1st term, $x_2^{\lambda_2}$ from 2nd, etc.: otherwise we will "add" some λ_j and λ_i in the exponent, making a bigger (lex) partition.
Another way to state Claim is that the $p(n) \times p(n)$ matrix M whose rows are the coeff's expressing p_λ in the basis m_μ is upper triangular (w/ nonzero entries on diag.)

$$M = \begin{pmatrix} \alpha^{\lambda^1}_{\mu^1} & \alpha^{\lambda^1}_{\mu^2} & \cdots & \\ 0 & \alpha^{\lambda^2}_{\mu^2} & & \\ & & \ddots & \\ 0 & & 0 & \alpha^{\lambda^m}_{\mu^m} \end{pmatrix}$$

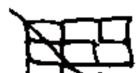
when we order rows/col's lexicographically.

Thus, in particular M is invertible so we can write m_μ as a sum of the p_λ. So the p_λ are a basis!
(they span $Sym(n)$ and there are the right # of them.)

To show $\{e_\lambda : \lambda \vdash n\}$ is a basis, we can do something similar, but now we need to use the transpose of our partitions.

(also called conjugate, sometimes denoted λ')

Recall the transpose of $\lambda = (\lambda_1, \ldots, \lambda_k)$ is what we get by reflecting Young diagram across main diagonal:

$\lambda = (3,3,1)$  $\Longleftrightarrow$  $\lambda^t = (3,2,2)$.

Now we... Claim $e_\lambda = \underbrace{\beta^{\lambda}_{\lambda^t}}_{\neq 0} m_{\lambda^t} + \sum_{\mu < \lambda^t} \beta^{\lambda}_{\mu} m_\mu$ for coeffs $\beta^{\lambda}_{\mu} \in \mathbb{C}$.

Why? consider expanding

$e_\lambda = (x_1 x_2 \cdots x_{\lambda_1} + \cdots)(x_1 x_2 \cdots x_{\lambda_2} + \cdots) \cdots (x_1 \cdots x_{\lambda_k} + \cdots)$

To make the biggest monomial here, we should take all terms of form $x_1 \cdots x_{\lambda_i}$ (so as many exponents "add" as possible). That product gives $x_1^{\lambda_1^t} x_2^{\lambda_2^t} \cdots$, so claim is proved. As before, implies transition matrix is invertible.

To prove $\{h_\lambda : \lambda \vdash n\}$ form a basis, we do something different. Namely, we consider g.f. product

$$\left(\sum_{k \geq 0} h_k(x_1, \ldots) t^k \cdot \sum_{k \geq 0} (-1)^k e_k(x_1, \ldots) t^k\right) = \prod_{i \geq 1} \frac{1}{1 - x_i t} \cdot \prod_{i \geq 1} 1 - x_i t = 1.$$

This says that $\sum_{k=0}^{n} h_k(x_1, \ldots) \cdot (-1)^{n-k} e_{n-k}(x_1, \ldots) = 0 \quad n \geq 1$.

By induction, this implies that e_n is a lin. comb. of products of h_k, $k \leq n$, i.e., that $e_n \in \mathrm{Span}_{\mathbb{C}}\{h_\lambda\}$, so in fact $e_\lambda \in \mathrm{Span}_{\mathbb{C}}\{h_\mu\}$ for all λ, so $\mathrm{Span}_{\mathbb{C}}\{h_\mu : \mu \vdash n\} = \mathrm{Sym}(n)$. ✓ ☒

Other important algebraic structures on Sym:

- A scalar product $\langle \cdot, \cdot \rangle : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathbb{C}$ given by $\langle m_\lambda, h_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$
- An involution $\omega : \mathrm{Sym} \to \mathrm{Sym}$ given by $\omega(h_\lambda) = e_\lambda$.
- A coproduct $\mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{Sym}$ which makes Sym into a Hopf algebra.

See Stanley for these! No time to discuss in this class!