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Schur ~~fun~~ functions: the "most important" basis of Sym

We now define the Schur functions $s_\lambda(x_1, x_2, \ldots)$, which are another basis of Sym – and (arguably) the most important one.
It's a bit hard to motivate what makes them so important:

i) W.r.t. the inner product $\langle \cdot, \cdot \rangle : \mathrm{Sym} \times \mathrm{Sym} \to \mathbb{R}$ just mentioned they are orthonormal: $\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise} \end{cases}$

ii) In the representation theory interpretations of Sym, they correspond to "irreducible representations."

The definition of $s_\lambda(x_1, x_2, \ldots)$ will be very different from other bases:

DEF'N Let $\underset{=(\lambda_1, \lambda_2, \ldots, \lambda_k)}{\lambda} \vdash n$ be a partition. Recall that λ's Young diagram has λ_i boxes in the i^{th} row:

$$\lambda = (3,3,1) \Leftrightarrow \lambda = \begin{array}{|c|c|c|} \hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad \text{(rows are left-justified)}$$

A semistandard Young tableau of shape λ is a filling of the boxes of its Young diagram w/ positive integers such that:

- entries are weakly increasing along rows
- entries are strictly increasing down columns

e.g. $T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ is a SSYT of shape $(3,3,1)$

For T a SSYT, content (T) is vector $co(T) := (c_1, c_2, \ldots) \in \mathbb{N}^\infty$ where $c_i = \#$ boxes w/ entry i

e.g. $co(T) = (2,3,1,1,0,0,\ldots)$

The Schur function $s_\lambda(x_1, x_2, \ldots)$ is then

$$s_\lambda := \sum_{\substack{T \text{ SSYT} \\ \text{of sh.} = \lambda}} \vec{x}^{\,co(T)} = \sum_{\substack{T \text{ SSYT} \\ sh(T) = \lambda}} \prod_{i \geq 1} x_i^{c_i(T)} \in \mathbb{C}[[x_1, x_2, \ldots]]$$

e.g. Let $\lambda = (2,1)$. Let's compute the Schur polynomial
$$s_\lambda(x_1, x_2, x_3) = s_\lambda(x_1, x_2, x_3, 0, 0, \ldots)$$
The SSYT of sh. $= (2,1)$ and entries in $\{1,2,3\}$ are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

So $s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 + 2x_1x_2x_3$
$$= 2m_{(1,1,1)}(x_1, x_2, x_3) + m_{(2,1)}(x_1, x_2, x_3)$$
a Symmetric polynomial! ← not a priori obvious it should be symmetric

In fact, $s_{(2,1)}(x_1, x_2, \ldots) = 2m_{(1,1,1)} + m_{(2,1)} \in \mathrm{Sym}$.

Schur functions generalize elementary + complete homo. sym. fn's:

Prop • $s_{(1^n)}(x_1, \ldots) = e_n(x_1, \ldots)$

• $s_{(n)}(x_1, \ldots) = h_n(x_1, \ldots)$

Pf:
$$s_{(1^n)}(x_1, \ldots) = s_{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}}(x_1, \ldots) = \sum_{T \text{ SSYT sh} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}} \vec{x}^{\,\mathrm{co}(T)}$$
But an SSYT of sh. $= \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$ is just $\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_n \\ \hline\end{array}$ w/ $i_1 < i_2 < \cdots < i_n$

So indeed $s_{(1^n)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} = e_n$.

Similarly $s_{(n)} = \sum_{T \text{ SSYT sh.} = \begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline\end{array}} \vec{x}^{\,\mathrm{co}(T)}$ and an SSYT of sh. $\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline\end{array}$ is $\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_n \\ \hline\end{array}$ w/ $i_1 \le \cdots \le i_n$

So $s_{(n)} = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n} = h_n$. ∎

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But for other shapes than single row/column, not clear that s_λ is symmetric. We prove this now.

<u>DEF'N</u> Let T be a SSYT. The i^{th} <u>Bender</u>-<u>Knuth</u> <u>involution</u> (for $i=1,2,\ldots$) applied to T, denoted $b_i(T)$, is the following operation:

- first, "<u>freeze</u>" all entries i ~~[illegible]~~ (directly) above an $i+1$, and all $i+1$'s (directly) below an i,
- then, in each row, if there are a unfrozen i's and b unfrozen $i+1$'s in this row, modify these entries so that there are b unfrozen i's and a unfrozen $i+1$'s (in unique way that preserves SSYT-ness).

<u>e.g.</u> Let's apply b_4 to

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline 1&1&1&1&2&2&2&3&3&3&4\\
\hline 2&2&3&3&3&3&\boxed{4}&\boxed{4}&4&5\\
\cline{1-10} 3&\boxed{4}&\boxed{4}&4&5&5&\boxed{5}&\boxed{5}\\
\cline{1-8} 5&\boxed{5}&\boxed{5}&6\\
\cline{1-4} 6&6\\
\cline{1-2}
\end{array}$$

$\square$ = frozen

1 unfrozen 4, 2 unfrozen 5's in 3rd row

$$b_4(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline 1&1&1&1&2&2&2&3&3&3&5\\
\hline 2&2&3&3&3&3&\boxed{4}&\boxed{4}&4&5\\
\cline{1-10} 3&\boxed{4}&\boxed{4}&4&4&5&\boxed{5}&\boxed{5}\\
\cline{1-8} 4&\boxed{5}&\boxed{5}&6\\
\cline{1-4} 6&6\\
\cline{1-2}
\end{array}$$

2 unfrozen 4's
1 unfrozen 5 in 3rd row,
etc...

<u>Prop.</u> $b_i(T)$ is an SSYT (of same shape as T!) with $\mathrm{co}(b_i(T)) = (i,i+1)\cdot \mathrm{co}(T)$, i.e., # i's in T = # $i+1$'s in $b_i(T)$ and vice-versa.

Also, $b_i(b_i(T)) = T$.

<u>Pf:</u> All statements are relatively straightforward. To see $\mathrm{co}(b_i(T)) = (i,i+1)\cdot \mathrm{co}(T)$, note that frozen i's + $i+1$'s come in pairs that cancel, while unfrozens get switched. $\blacksquare$

Cor For any λ, s_λ is a symmetric function.

Pf: Bender-Knuth involutions show that $(i,i+1)\cdot s_\lambda = s_\lambda$ (since $\sum_{T\ \mathrm{SSYT}\ sh(T)=\lambda} \vec{x}^{\,co(T)} = \sum_{T:sh(T)=\lambda} \vec{x}^{\,co(b_i(T))} = \sum_{T:sh(T)=\lambda} \vec{x}^{\,(i,i+1)\cdot co(T)} = (i,i+1)\cdot \sum_{T:sh(T)=\lambda} \vec{x}^{\,co(T)}$.)

But then note that any permutation $\sigma \in S_n$ is a composition of adjacent transpositions $\sigma = (i_1, i_1+1)\cdot(i_2,i_2+1)\cdots(i_\ell,i_\ell+1)$

(Think about sorting ~~a~~ numbers in a line: $7\,1\,3\,2\,5\,6\,4$, can always do it by swapping adjacent positions.)

So $\sigma \cdot s_\lambda = s_\lambda$ for any $\sigma\in S_n$, so s_λ is symmetric! $\square$

Thm $\{s_\lambda : \lambda \vdash n\}$ is a basis of $\mathrm{Sym}(n)$.

Pf: Just proved that s_λ for $\lambda\vdash n$ is symmetric, and that it has degree n is clear. Since there are correct # of s_λ for a basis, what we need to show is that they span all of $\mathrm{Sym}(n)$.

We do this, like with the other bases, by a triangularity argument. So write

$$s_\lambda = \sum_\mu K_{\lambda,\mu}\, m_\mu .$$

Note that $K_{\lambda,\mu} := \#$ SSYT w/ $sh=\lambda$ and $co=\mu$.

We claim that $K_{\lambda,\mu}\neq 0 \Rightarrow \mu \leq \lambda$ in lex. order.

Indeed, there is 1 tableau counted by $K_{\lambda,\lambda}=1$: we have all i's in the i^{th} row

e.g., $\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2\\ \cline{1-3} 3&3&3\\ \cline{1-3} 4\\ \cline{1-1}\end{array}$

Now suppose $\mu\neq\lambda$ and $K_{\lambda,\mu}\neq 0$. Let j be smallest # s.t. $\mu_j \neq \lambda_j$. Then $\lambda_i = \mu_i\ \forall i<j$. So a tableau counted by $K_{\lambda,\mu}$ has all i's in row i for $i<j$. So j^{th} row has $<\lambda_j$ j's $\Rightarrow \mu < \lambda$. ✓ $\square$

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Expanding Schur functions in the other bases

From what we just explained, we have

$$s_\lambda = \sum_\mu K_{\lambda,\mu}\, m_\mu$$

where $K_{\lambda\mu} := \#\{\text{SSYT } T : sh(T)=\lambda,\ co(T)=\mu\}$
← called the Kostka numbers

But we also have the e_μ, h_μ, and p_μ bases, so can ask what s_λ looks like in these bases.

The expansion of Schurs into power sums:

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi^\lambda(\mu)\, p_\mu \quad ; \; z_\mu = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots \text{ if } \mu = 1^{m_1} 2^{m_2} \cdots$$

is perhaps the most important one, because

$\chi^\lambda(\mu)$ = "character of irreducible representation of S_n indexed by $\lambda \vdash n$ at permutation of cycle type μ."

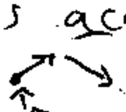
There is a combinatorial formula for $\chi^\lambda(\mu)$ called the "Murnaghan-Nakayama rule"; see Ch. 7 Stanley EC2

But it's beyond scope of this class...

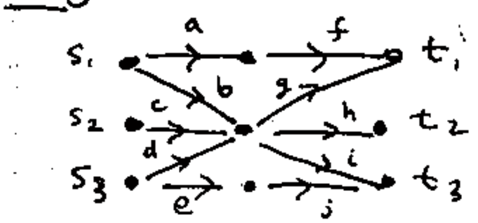
Instead we'll focus on the expansion of s_λ into e_μ / h_μ.

The formula for writing s_λ in the e_μ's/h_μ's is called the Jacobi-Trudi formula and it expresses s_λ as a determinant.

To prove the J-T formula, we will apply another result: the Lindström-Gessel-Viennot lemma which is itself a very powerful enumerative tool worth knowing about.

DEF'N A directed graph (or digraph) $G=(V,E)$ has vertex set V and directed edge set E, where a directed edge $e=(u,v)$ is an ordered pair of vertices we draw as an arrow: $u \to v$. We say G is acyclic if it has no directed cycles: . An (acyclic) network is an acyclic digraph w/ distinguished source vertices $s_1, s_2, \ldots, s_n$ and target vertices $t_1, \ldots, t_n$, and a weight function $wt: E \to \mathbb{R}$ on edges.

e.g. Here is an acyclic network w/ 3 sources + targets:



w/ $a,b,c,d,e,f,g,h,i,j \in \mathbb{R}$ as edge weights.

DEF'N A path P in a digraph is a sequence of edges $s \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \cdots \bullet \xrightarrow{e_n} t$ connecting s to t. We define the weight of P to be $wt(e_1) \cdot \cdots \cdot wt(e_n)$.

The path matrix of network G is $n \times n$ matrix M with $M_{i,j} := \sum_{\substack{\text{paths} \\ P: s_i \text{ to } t_j}} wt(P)$.

To a tuple $T = (P_1, \ldots, P_n)$ of paths we associate weight $wt(T) = wt(P_1) \cdot \cdots \cdot wt(P_n)$. We say the tuple is nonintersecting if all the vertices in P_i and P_j are disjoint, for every $i \neq j$.

Thm (Lindström-Gessel-Viennot Lemma)
Let M be the path matrix of acyclic network G.
Then $\det(M) = \sum_{\substack{\text{non-intersecting tuple} \\ T=(P_1,\dots,P_n) \\ P_i : s_i \to t_{\sigma(i)}}} \operatorname{sgn}(\sigma)\cdot \operatorname{wt}(T)$.

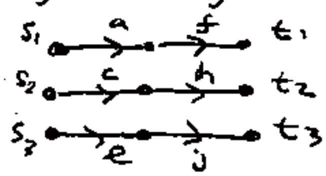
(Recall for a permutation $\sigma \in S_n$, $\operatorname{sgn}(\sigma) = (-1)^{\#\text{ inversions}(\sigma)}$.)

<u>e.g.</u> W/ G the network from previous example

path matrix is
$$M = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ \hline s_1 & af+bg & bh & bi \\ s_2 & cg & ch & ci \\ s_3 & dg & dh & di+ej \end{array}$$

and $\det(M) = \big((af+bg)(ch)(di+ej) + (bh)(ci)(dg) + (bi)(cg)(dh)\big)$
$\quad - \big((bi)(ch)(dg) + (bh)(cg)(di+ej) + (af+bg)(ci)(dh)\big)$

$= (af)(ch)(ej)$ ← wt of unique tuple of non-int. lattice paths



<u>NOTE</u>: $\operatorname{sgn}\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = +1$ ✓

In this example, we see a very important special case:

<u>Cor</u> (Planar LGV lemma)
Suppose acyclic network G is <u>planar</u> (i.e., edges only cross at vertices) drawn in a disc w/ sources $s_1,\dots,s_n$ and targets $t_n,\dots,t_1$ on boundary (in counterclockwise order), like

Then,
$$\det(M) = \sum_{\substack{\text{non-intersecting} \\ T=(P_1,\dots,P_n) \\ P_i : s_i \to t_i}} \operatorname{wt}(T).$$

//

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Pf of LGV Lemma: (from Ch. 2 of Sagan)

We will use a technique from last semester: Sign-reversing involution.

First, we use the "Leibniz formula" for determinant:

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \geq 1} M_{i,\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \geq 1} \sum_{\substack{\text{path} \\ P: s_i \to t_{\sigma(i)}}} wt(P)$$

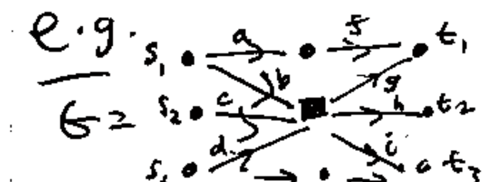
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{\substack{\text{tuple of paths} \\ T = (P_1, \ldots, P_n) \\ P_i: s_i \to t_{\sigma(i)}}} wt(T)$$

$$= \sum_{\substack{\text{tuple of paths} \\ T = (P_1, \ldots, P_n),\ P_i: s_i \to t_{\sigma(i)}}} \operatorname{sgn}(\sigma) \cdot wt(T)$$

So $\det(M)$ is naturally the generating function of all tuples of paths connecting sources to sinks.

To show that this sum can be taken over only the non-intersecting tuples, we will cancel all the intersecting tuples, by defining an appropriate sign-reversing involution:

- given an intersecting tuple $T = (P_1, \ldots, P_n)$, let (i,j) be lex. smallest pair such that $P_i + P_j$ intersect, and let v be the last vertex they intersect at. Define P_i' to be P_i up to v, and P_j after that and P_j' to be P_j up to v, and P_i after that. Set $T' := (P_1, P_2, \ldots, P_i', \ldots, P_j', \ldots, P_n)$.

e.g. $G =$ 

$T = \big(\underbrace{\xrightarrow{b}\blacksquare\xrightarrow{g}}_{P_1 =},\ \underbrace{\xrightarrow{c}\blacksquare\xrightarrow{h}}_{P_2 =},\ \underbrace{\xrightarrow{d}\blacksquare\xrightarrow{j}}_{P_3 =} \big)$

$T' = \big(\underbrace{\xrightarrow{b}\blacksquare\xrightarrow{h}}_{P_1' =},\ \underbrace{\xrightarrow{c}\blacksquare\xrightarrow{g}}_{P_2' =},\ \underbrace{\xrightarrow{d}\blacksquare\xrightarrow{j}}_{P_3} \big)$

swapped ending of $P_1 + P_2$

Then $T \mapsto T'$ is an involution, and if σ and σ' are the permutations corresponding to T, T' we have $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma')$ because $\sigma' = \sigma\cdot(i,j)$ ~~$(ij)\sigma$~~.
[Exercise: check this fact about permutation signs.]

Also, T and T' use same edges, so $\mathrm{wt}(T) = \mathrm{wt}(T')$.

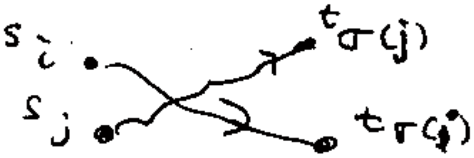
Thus $T \mapsto T'$ is a sign-reversing involution on all intersecting tuples, so the intersecting tuples cancel in the sum and we get

$$\det(M) = \sum_{\substack{\text{tuples}\\ T=(P_1,\dots,P_n):\, P_i : s_i \to t_{\sigma(i)}}} \mathrm{sgn}(\sigma)\cdot \mathrm{wt}(T) = \sum_{\substack{\text{non-intersecting}\\ T=(P_1,\dots,P_n):\, P_i : s_i \to t_{\sigma(i)}}} \mathrm{sgn}(\sigma)\cdot \mathrm{wt}(T) \quad \blacksquare$$

<u>Pf of planar LGV corollary</u>:

If G looks like  and is planar,

then for any $T = (P_1, \dots, P_n)$ whose σ is not $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ we will have an intersection: there will be some $i < j$ with $\sigma(i) > \sigma(j)$.



So for planar networks like this, we only need to sum over T's with $\sigma = $ identity, which have $\mathrm{sgn}(\sigma) = +1$. $\blacksquare$

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The Jacobi-Trudi formulas

We will now use LGV lemma to prove.

Thm (Jacobi-Trudi) For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$,

(a) $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le k}$

(b) $s_\lambda = \det\left(e_{\lambda_i - i + j}\right)_{1 \le i,j \le k}$

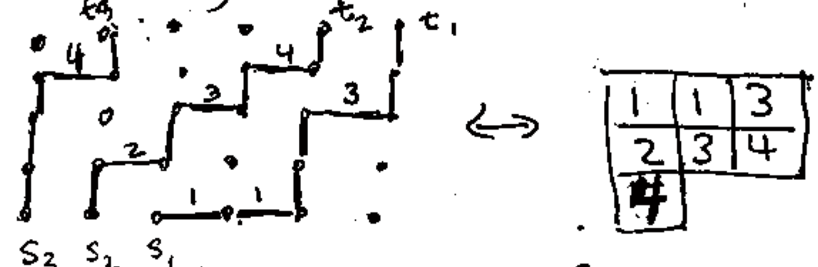
<u>eg.</u> $s_{(2,1)} = \det\begin{bmatrix} h_2 & h_3 \\ h_0 & h_1 \end{bmatrix} = \overset{m_{(3)} + 2m_{(2,1)} + 3m_{(1,1,1)}}{\overset{\shortparallel}{h_2 h_1}} - \overset{m_{(3)} + m_{(2,1)} + m_{(1,1,1)}}{\overset{\shortparallel}{h_3}} \cdot 1 = m_{(2,1)} + 2m_{(1,1,1)}$ ✓

<u>NOTE</u> $h_r = e_r = 0$ for $r < 0$ in this formula.

<u>Pf of Jacobi-Trudi:</u> We prove (a); (b) is similar.

Construct the following network G based on λ:

e.g. $k = 3$
$\lambda = (3,3,1)$



The network is a part of the grid $\mathbb{Z}^2$, w/ all edges directed right and up. Our sources are $s_i = (k-i, 1)$ and targets are $t_i = (k - i + \lambda_i, n)$ for $i = 1, 2, \ldots, k$. (Here n is a number will will send $\longrightarrow \infty$.)

As depicted above, tuples $(P_1, \ldots, P_k)$ of non-intersecting lattice paths w/ $P_i : s_i \to t_i$ correspond bijectively to SSYT of shape $= \lambda$

(w/ entries $\leq n$): the corresponding tableau T has entries $i_1, \ldots, i_{\lambda_j}$ in the j^{th} row, where $i_1, \ldots, i_{\lambda_j}$ are the horizontal step heights of path P_j.

There is something to check here: that the non-intersectingness of the paths is equivalent to the SSYT condition. That's an exercise for you...

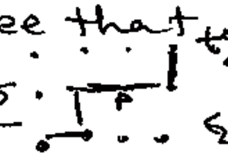
This bijection tells us what the edge weights of our network should be: vertical steps have wt $= 1$, and a horizontal step at height i has wt $= x_i$.

Thus the LGV Lemma applied to this network says

$$S_\lambda(x_1, \ldots, x_n) = \det(M), \text{ where } M_{i,j} = \sum_{\substack{\text{paths} \\ p:\, s_i \to t_j}} \mathrm{wt}(p).$$

But it's not hard to see that

$$\sum_{\text{paths } P:\, s_i \to t_j} \mathrm{wt}(P) = \sum \quad t_j = (k-j+\lambda_j, n)$$



$s_i = (k-i, 1)$

$$= h_{\lambda_j - j + i}(x_1, \ldots, x_n)$$

choose any size $\lambda_j - j + i$ multiset of horiz. heights of steps.

So $S_\lambda(x_1, \ldots, x_n) = \det\left(h_{\lambda_j - j + i}\right) \underset{\text{transpose!}}{=} \det\left(h_{\lambda_i - i + j}\right)$,

and we get the J-T formula in limit $n \to \infty$. ∎

Rmk: Can use J-T formula (+ some determinant manipulation) to show

$$S_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_j^{\lambda_j + n - i}\right)}{\det\left(x_j^{n-i}\right)}$$

"Vandermonde determinant" $= \prod_{1 \leq j < i \leq n} (x_j - x_i)$

which is actually the original definition of the Schur polynomials from late 19^{th}/early 20^{th} century "Bialternant definition"