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Specializations of symmetric functions

So far we haven't done much counting w/ symmetric fn's. One way to get interesting sequences of #'s from sym fn's is by specializing them, i.e., plugging in values.

Prop: (a) $e_k(\overbrace{1,1,\ldots,1}^{n},0,0,\ldots) = \binom{n}{k}$

(b) $h_k(\overbrace{1,1,\ldots,1}^{n}) = \left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ ← "multichoose"

Pf: Recall $e_k(x_1,\ldots,x_n) = \sum_{S\subseteq[n],\,\#S=k} \prod_{i\in S} x_i$ ← product of k distinct variables, so clearly setting $x_i = 1$ $\forall\, 1\le i\le n$ gives $e_k(1,1,\ldots,1) = \binom{n}{k}$.

Similarly, $h_k(\overbrace{1,1,\ldots,1}^{n}) = \left(\!\binom{n}{k}\!\right) = \#$ k-multisets of $[n] = \{1,2,\ldots,n\}$

Recall that from "stars and bars" we showed that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ e.g. $\{1,1,3,4,4\} \subseteq [5]$ ← multi-subset

$\Leftrightarrow$ $**\,|\,|\,*\,|\,**\,|$
1's 2's 3's 4's 5's

$\square$

In fact, we can similarly get the q-binomials:

DEF'N The q-binomial coefficient $\begin{bmatrix} a+b \\ b \end{bmatrix}_q$ is the g.f.

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q := \sum_{\lambda\subseteq a\times b} q^{|\lambda|}$$ of partitions in an $a\times b$ rectangle.

e.g. $\begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_q = q^4+q^3+2q^2+q+1$ since $\emptyset$, ... $\subseteq$ ...

We showed last semester that
$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \frac{[a+b]_q!}{[a]_q!\,[b]_q!}, \text{ where}$$

"q-number" $\qquad\qquad\qquad\qquad$ "q-factorial"

$$[n]_q = 1+q+\cdots+q^{n-1} = \frac{(1-q^n)}{(1-q)} \quad \text{and} \quad [n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q$$

e.g. $\begin{bmatrix}2+2\\2\end{bmatrix}_q = \dfrac{[4][3]\cancel{[2]}\cancel{[1]}}{[2][1]\cancel{[2]}\cancel{[1]}} = (1+q+q^2)\cdot\dfrac{(1-q^4)}{(1-q^2)} = (1+q+q^2)(1+q^2)$ ✓

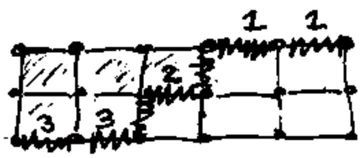
Thm (a) $e_k(1,q,\dots,q^{n-1}) = q^{\binom{k}{2}} \cdot \begin{bmatrix}n\\k\end{bmatrix}_q$

(b) $h_k(1,q,\dots,q^{n-1}) = \begin{bmatrix}n+k-1\\k\end{bmatrix}_q$

Pf: We do (b) first. Observe that

$$h_k(1,q,\dots,q^{n-1}) = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} q^{\sum_{j=0}^{k}(i_j-1)}.$$

To any such k-multiset $1\le i_1\le\cdots\le i_k\le n$, let's associate a partition λ inside the $(n-1)\times k$ rectangle, as follows:

e.g. $n=3$, $k=5$ $\lambda =$ 

$\leftrightarrow$ size 5 multiset $\{1,1,2,3,3\} \subseteq [3]$

i.e., the values of the multiset tell us heights of horizontal steps on SE border of partition λ (where ht 1 = top and ht n = bottom)

Under this correspondence, $|\lambda| = \sum_{j=0}^{k}(i_j-1)$ (number of boxes).

So indeed $h_k(1,q,\dots,q^{n-1}) = \begin{bmatrix}n-1+k\\k\end{bmatrix}_q$. ✓

For (a): similar. Can use trick of changing k-subset $i_1<i_2<\cdots<i_k$ to k-multisubset $i_1<i_2-1<\cdots<i_k-(k-1)$.
The difference $1+2+\cdots+(k-1) = \binom{k}{2}$ explains factor of $q^{\binom{k}{2}}$.

Cor ("Principal specialization" of e_k and h_k)

(a) $e_k(1,q,q^2,\dots) = q^{\binom{k}{2}} \cdot 1/(1-q)(1-q^2)\cdots(1-q^k)$

(b) $h_k(1,q,q^2,\dots) = 1/(1-q)(1-q^2)\cdots(1-q^k)$.

Pf: Note $\displaystyle\lim_{n\to\infty}\begin{bmatrix}n\\k\end{bmatrix}_q = \lim_{n\to\infty}\frac{[n]_q!}{[n-k]_q!\,[k]_q!} = \lim_{n\to\infty}\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$

$= 1/(1-q)(1-q^2)\cdots(1-q^k)$. ■

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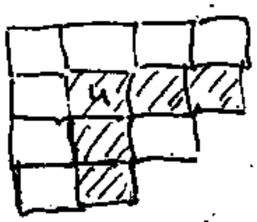
Principal specialization of Schur functions

We could ask about specialization of other sym. fn's, like p_k's or m_λ's. ← HW #21. But now we discuss s_λ's:

DEF'N Let λ be a partition, viewed as a Young diagram, and let $u \in \lambda$ be a box of the Young diagram. The hook of u is all boxes below or to the right of u, together with u itself:

e.g.



boxes in ▨ = hook $h(u) = 5$

The hook length $h(u) := \#$ boxes in hook of u.

Thm (Principal specialization of Schur function)
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. Then,

$$s_\lambda(1, q, q^2, \ldots) = q^{b(\lambda)} \cdot \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}}$$

where $b(\lambda) = 0 \cdot \lambda_1 + 1 \cdot \lambda_2 + 2 \cdot \lambda_3 + \cdots = \sum_{i=1}^{k} (i-1) \cdot \lambda_i$.

e.g. $e_k(1, q, q^2, \ldots) = s_{(1^k)}(1, q, \ldots) = q^{\binom{k}{2}} \cdot \frac{1}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$

since hook lengths are $\begin{array}{|c|}\hline k \\ \hline k-1 \\ \hline \vdots \\ \hline 1 \\ \hline\end{array}$ for single column.

Similarly, $h_k(1, q, \ldots) = s_{(k)}(1, q, \ldots) = \frac{1}{(1-q^k)\cdots(1-q)}$

since $\begin{array}{|c|c|c|c|}\hline k & k-1 & \cdots & 1 \\ \hline\end{array}$ are hook lengths for single row.

We saw these cases last class.

e.g. Let $\lambda = (2,1)$. Then

$$s_\lambda(1,q,q^2,\dots) = \sum_{\text{SSYT } T,\ sh(T)=\lambda} q^{\text{sum of entries of } T - |\lambda|}$$

$$= \underset{\begin{smallmatrix}1&1\\2&\end{smallmatrix}}{q} + \underset{\begin{smallmatrix}1&1\\3&\end{smallmatrix}\ \begin{smallmatrix}1&2\\2&\end{smallmatrix}}{2q^2} + \underset{\begin{smallmatrix}1&1\\4&\end{smallmatrix}\ \begin{smallmatrix}1&2\\3&\end{smallmatrix}\ \begin{smallmatrix}1&3\\2&\end{smallmatrix}}{3q^3} + \underset{\begin{smallmatrix}1&1\\5&\end{smallmatrix}\ \begin{smallmatrix}1&2\\4&\end{smallmatrix}\ \begin{smallmatrix}1&3\\3&\end{smallmatrix}\ \begin{smallmatrix}1&4\\2&\end{smallmatrix}\ \begin{smallmatrix}2&2\\3&\end{smallmatrix}}{5q^4} + \cdots$$

$$= q \cdot \frac{1}{(1-q)^2(1-q^3)} = q^{b(\lambda)} \cdot \prod_{u\in\lambda} \frac{1}{1-q^{h(u)}}$$ since hook lengths: $\begin{smallmatrix}3&1\\1&\end{smallmatrix}$.

In fact, even have a "finite version":

Thm (Stanley's hook-content formula)

$$s_\lambda(1,q,q^2,\dots,q^{n-1}) = q^{b(\lambda)} \cdot \prod_{u\in\lambda} \frac{1-q^{c(u)+n}}{1-q^{h(u)}},$$

where $c(u) := j - i$ for box $u = (i,j)$.

As before can get principal specialization via limit $n \to \infty$.

Pf sketch: Starts w/ the "bialternant formula"

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det\left(x_j^{\lambda_i + n - i}\right)}{\prod_{1\le i<j\le n}(x_j - x_i)},$$ makes

substitution $x_i \to q^{i-1}$, does some algebraic manipulations of the determinant. (See Stanley EC2). ☒

3/21 We'd prefer a combinatorial proof, which we will give for the principal specialization. Starting point:

DEF'N A reverse plane partition of shape λ is a filling of the boxes of λ with nonnegative integers that is weakly increasing in both rows and columns.

e.g. $\begin{smallmatrix}0&0&2\\0&1&3\\1&1&\end{smallmatrix}$ is an r.p.p. of sh $= (3,3,2)$.

Let $RPP(\lambda) :=$ set of r.p.p.'s of sh $= \lambda$. There is a simple bijection $\phi: SSYT(\lambda) \to RPP(\lambda)$ that subtracts i from all boxes in i^{th} row:

e.g. $\phi\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 5 \\ \cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 & 1 \\ \cline{1-2} 0 & 2 \\ \cline{1-2}\end{array}$

Notice that via this bijection, sum of entries in T $= b(\lambda) + |\lambda| +$ sum of entries in $\phi(T)$.

So principal specialization of s_λ is equivalent to ...

Thm $$\sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \prod_{u \in \lambda} \frac{1}{1-q^{h(u)}}$$

where $|\pi| :=$ sum of entries of r.p.p. π.

We will explain a bijective proof of this thm...

To prove thm, it is enough to construct a bijection

$\phi: RPP(\lambda) \to \{$arbitrary $\mathbb{N}$-fillings A of boxes of $\lambda\}$

s.t. sum of entries of π $= \underbrace{\sum_{u \in \lambda} A(u) \cdot h(u)}_{=: wt(A)}$ for $A = \phi(\pi)$

Why? Because then:

$$\sum_{\pi \in RPP(\lambda)} q^{|\pi|} = \sum_{A \text{ an } \mathbb{N}\text{-filling of } \lambda} q^{wt(A)} = \sum_{A} \prod_{u \in \lambda} q^{A(u) \cdot h(u)}$$

$$= \prod_{u \in \lambda} \left(1 + q^{h(u)} + q^{2 \cdot h(u)} + \cdots\right) = \prod_{u \in \lambda} \frac{1}{1-q^{h(u)}} \quad \checkmark$$

choose each value $A(u)$ independently

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The bijection ϕ is called the <u>Hillman-Grassl algorithm</u>. It is defined via a series of steps. We start off by writing our $\pi \in RPP(\lambda)$ next to the all 0's filling:

$$RPP(\lambda) \ni \pi = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & \\ \cline{1-3} 3 & & & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \\ \cline{1-3} 0 & & & \\ \cline{1-1}\end{array} = A_0$$

$\lambda = (4,3,1)$, $\pi = \pi_0$

Then we find a path of boxes in π ($=\pi_0$) as follows:

- Start at northeastern most box (i,j) for which $\pi(i,j) \neq 0$,
- if we're at box (i,j), then
  - move to $(i, j-1)$ if $\pi(i,j) = \pi(i,j-1)$
  - move to $(i+1, j)$ otherwise
- repeat the previous until we exit λ (by leaving south out of a column).

For example, w/ the above example we get this path

$$\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 3 & & & \end{array}$$

$\leftarrow$ enter at row $1 = i'$

exit at column $1 = j'$

we go left when value is same as where we are, otherwise go down.

Then we define π_1 by subtracting 1 from all boxes on the path, (from π) and we define A_1 by adding 1 to A_0 in position (i', j'), where i' = row we entered at, and j' = column we exited at

Thus,
$$\pi_1 = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & \\ 2 & & & \end{array} \qquad A_1 = \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & & & \end{array} \leftarrow \text{added 1 to } (1,1)$$

Notice that the # of boxes in the path must be the same as the number of boxes in the hook of (i', j'):

$$\begin{array}{ccc} & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array} = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & & \\ \bullet & & \end{array}$$

because the path is a "ribbon"

So we have that $|\pi_0| - |\pi_1| = wt(A_1) - wt(A_0)$.

Then we repeat: find a path in π_1 using the same rules, and define π_2 and A_2 from π_1 and A_1 in same way:

$$\pi_1 = \begin{array}{cccc} 1&1&1&1 \\ 2&2&3& \\ 2&&& \end{array} \quad A_1 = \begin{array}{cccc} 1&0&0&0 \\ 0&0&0& \\ 0&&& \end{array} \Rightarrow \pi_2 = \begin{array}{cccc} 0&0&0&0 \\ 1&2&\boxed{3}& \\ 1&&& \end{array} \quad A_2 = \begin{array}{cccc} 2&0&0&0 \\ 0&0&0& \\ 0&&& \end{array}$$

$$\Rightarrow \pi_3 = \begin{array}{cccc} 0&0&0&0 \\ 1&2&2& \\ 1&&& \end{array} \quad A_3 = \begin{array}{cccc} 2&0&0&0 \\ 0&0&1& \\ 0&&& \end{array} \Rightarrow \pi_4 = \begin{array}{cccc} 0&0&0&0 \\ 1&1&1& \\ 1&&& \end{array} \quad A_4 = \begin{array}{cccc} 2&0&0&0 \\ 0&1&1& \\ 0&&& \end{array}$$

$$\Rightarrow \pi_5 = \begin{array}{cccc} 0&0&0&0 \\ 0&0&0& \\ 0&&& \end{array} \quad \underset{\phi(\pi)}{\overset{\shortparallel}{A_5}} = \begin{array}{cccc} 2&0&0&0 \\ 1&1&1& \\ 0&&& \end{array}$$

We stop when we reach π_k being all 0's. Then we set $\phi(\pi) := A_k$.

Note that $|\pi| = (|\pi_0| - |\pi_1|) + (|\pi_1| - |\pi_2|) + \cdots + (|\pi_{k-1}| - \underbrace{|\pi_k|}_{0})$

$$= (wt(A_k) - wt(A_{k-1})) + \cdots + (wt(A_1) - \underset{0}{\underset{\shortparallel}{wt(A_0)}}) = wt(A_k) = wt(\phi(\pi))$$

So indeed we defined map $\phi : RPP(\lambda) \to \{\mathbb{N}\text{-fillings } A\}$ which has the correct behavior on the weights of the fillings.

Need to check that ϕ is a <u>bijection</u>. To do that, we will show that is <u>invertible</u>, i.e., that we can undo the steps.

To explain inverse procedure:

1) Note that if we increment (i_1', j_1') before (i_2', j_2'), then either $i_1' < i_2'$, or $i_1' = i_2'$ and $j_1' \geq j_2'$.

   This tells us the <u>reverse</u> order to <u>decrement</u> values of A in the ~~reverse~~ inverse procedure

2) Show that we can build reverse of any path by <u>entering</u> at bottom of column j', and moving <u>right</u> when entry to the right is same, otherwise move <u>up</u> (stopping when we reach right of row i').

For the details of this proof of bijectivity, see Sagan.

Main takeaway: we can "locally" reverse the steps. ∎