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Longest increasing subsequences

<u>DEF'N</u> Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ be a permutation. A <u>subsequence</u> of σ is $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ for $i_1 < \cdots < i_k$ and is <u>increasing</u> if $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}$.

Let $\mathrm{lis}(\sigma) :=$ length of <u>longest increasing subsequence</u>

e.g. For $\sigma = \underline{2}\ \underline{4}\ \underline{7}\ 9\ \underline{5}\ 1\ 3\ \underline{6}\ \underline{8}$ have $\mathrm{lis}(\sigma) = 5$
with longest ~~increasing~~ increasing subsequence underlined.

<u>Note</u>! L.I.S. need not be unique : $\underline{1}\ \underline{2}\ \underline{4}\ \underline{3}$

● Increasing subsequences are a basic kind of <u>permutation pattern</u> (ask Prof. Burstein for more info...) Studying LIS's is very natural from point of view of <u>statistical</u> analysis of <u>time series</u> data.

There is a close connection between the Robinson-Schensted algorithm and longest increasing subsequences:

<u>Thm</u> Suppose $\sigma \xmapsto{RS} (P, Q)$ w/ $\mathrm{sh}(P) = \lambda = (\lambda_1, \lambda_2, \cdots)$. Then $\lambda_1 = \mathrm{lis}(\sigma)$.

e.g. $\sigma = 5\ \underline{2}\ \underline{3}\ 6\ \underline{4}\ 1\ \underline{7} \xmapsto{RS} \left(P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}\ ,\ Q = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \right)$

● and indeed $\lambda_1 = 4 = \mathrm{lis}(\sigma)$.

But <u>note</u>: 1st row of P ($= 1\ 3\ 4\ 7$) is <u>not</u>
        a LIS of σ (just has same length)

<u>Pf of thm</u>: Suppose $\emptyset = P_0, P_1, \ldots, P_n = P$ is the sequence of insertion tableaux we build up when inserting $\sigma_1, \sigma_2, \ldots, \sigma_n$.

  <u>Claim</u>: When inserting σ_k into P_{k-1}, if it enters in the $j^{\underline{th}}$ column, then the longest increasing subsequence ending at σ_k has length j.

<u>Pf</u>: By induction. The case $k=1$ is fine. So suppose x is entry in P_{k-1} in position $(1, j-1)$ (i.e., left of σ_k). Then by induction there is a subsequence σ' of $\sigma_1, \ldots, \sigma_{k-1}$ of length $j-1$ ending at x, and since $x < \sigma_k$ (or else we would've bumped it), the concatenation $\sigma' \sigma_k$ is a length j increasing subsequence. Similarly, to show there cannot be a longer subsequence, let $y \in \{\sigma_1, \ldots, \sigma_{k-1}\}$ be s.t. $y < \sigma_k$. By induction, when we inserted y we did so at col. with longest subseq. ending at y, call it j'. Cannot have $j' \leq j$, otherwise we would've inserted σ_k into a later column. So $j' < j$, and so longest inc. subseq. ending at σ_k can have length at most $j' + 1 \leq j$. ✓   ▨

$\overline{\text{What}}$ about the whole shape $\lambda = (\lambda_1, \lambda_2, \ldots)$?

<u>Thm</u>(Greene) Suppose $\sigma \overset{\text{RSK}}{\mapsto} (P, Q)$ w/ $sh(P) = \lambda$. Then for all k, $\lambda_1 + \lambda_2 + \cdots + \lambda_k =$ length of

longest subsequence of σ
that is a <u>union</u> of k <u>increasing</u>
<u>subsequences</u>.

$\underline{e.g.}$ w/ $\sigma = 2\underline{4}\underline{7}\underline{9}51368$, have $P = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \hline 7 \\ \hline \end{array}$ and

$k=2$

$5+3=8$ ✓

$2479 \sqcup 1368$ is a union of 2 increasing subsequences.

4/15 Can define <u>decreasing subsequences</u> of perm. σ analogously, and let $lds(\sigma) :=$ length of longest decr. subseq.

<u>Thm</u> If $\sigma \overset{RS}{\longmapsto} (P, Q)$ w/ $sh(P) = \lambda$, then $lds(\sigma) = \ell(\lambda)$

length of λ

$(= \lambda_1^t)$

In fact, this follows immediately from...

<u>Thm</u>* For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ let $\sigma^{rev} = \sigma_n \sigma_{n-1} \cdots \sigma_1$. Then if $\sigma \overset{RS}{\longmapsto} (P, Q)$ have $\sigma^{rev} \longmapsto (P', Q')$ where

$P' = P^t \leftarrow$ <u>transpose</u>.

$\overline{To}$ prove this symmetry property of RS, can use <u>column insertion</u>, which works same as (row) insertion, but where we try to put # into 1st column, and bump #'s from i^{th} column to $(i+1)^{st}$ column, etc.

<u>Key Lemma</u> Row and column insertions commute, i.e.,

$T \xrightarrow{row} a \xleftarrow{col} b = T \xleftarrow{col} b \xleftarrow{row} a$.

Pf: See Sagan. ∎

<u>Pf of thm</u>*: $P' = \sigma_1 \xrightarrow{row} \cdots \sigma_{n-1} \xrightarrow{row} \sigma_n \xrightarrow{row} \emptyset$ (1st insertion is same w/ row or col)

$= \sigma_1 \xrightarrow{row} \cdots \sigma_{n-1} \xrightarrow{row} \sigma_n \xrightarrow{col} \emptyset$

$= \sigma_n \xrightarrow{col} \sigma_1 \xrightarrow{row} \cdots \sigma_{n-1} \xrightarrow{row} \emptyset$ (key lemma)

$= \sigma_n \xrightarrow{col} \sigma_{n-1} \xrightarrow{col} \cdots \sigma_1 \xrightarrow{col} \emptyset$ (repeat)

$= \left(\sigma_n \xrightarrow{row} \sigma_{n-1} \xrightarrow{row} \cdots \sigma_1 \xrightarrow{row} \emptyset \right)^t$ (transpose of col insert = row insert)

$= P^t$. ✓ ∎

Cor (Erdős-Szekeres Theorem)
For any $\sigma \in S_{(n-1)(m-1)+1}$, have either
$lis(\sigma) \geq n$ or $lds(\sigma) \geq m$.

Pf: Best way to minimize width and length of a partition
is $\lambda = {}_{m-1}\overbrace{\boxed{}}^{n-1}$ but we need one more box ✓ 📦

Q: What is the **expected** length of longest incr. subseq.
of a **random** permutation?

Let $X_n := lis(\sigma)$ for $\sigma \in S_n$ (uniformly) random.

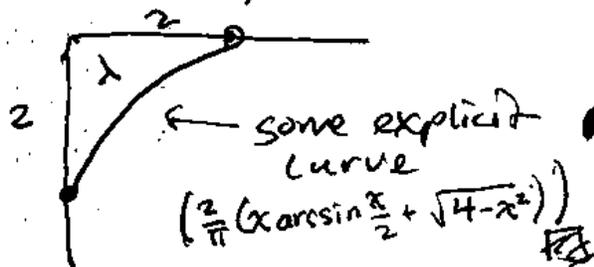**Ulam's Problem**: Compute $\lim\limits_{n\to\infty} \dfrac{\mathbb{E}\, X_n}{\sqrt{n}} = c$. ●
c. 1960's

E-S Thm says for any $\sigma \in S_n$, have $lis(\sigma) \geq \sqrt{n}$ or
$lis(\sigma^{rev}) \geq \sqrt{n}$

So that $c \geq \frac{1}{2}$. In fact...

Thm (Logan-Shepp, Kerov Vershik, 1977)
Solution to Ulam's Problem is $c = 2$.

Idea of pf: Same as asking for length of λ when we
insert $\sigma \in S_n$ into RS. In fact, this random
partition λ has
a precise
limit shape
(rescaling by $\frac{1}{\sqrt{n}}$):

← some explicit
curve
$\left(\frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right)\right)$ 📦

4/18

Representation Theory of finite Groups:

In the last couple days, I want to explain why ring of sym. fn.'s is important in algebra...

DEF'N Let V be an n-dim'l vector space over $\mathbb{C}$.

The general linear group $GL(V) = \{$ invertible linear maps $V \to V \}$.

I.e., $GL(V) \cong \{ n \times n \; \mathbb{C}$-matrices M w/ $\det(M) \neq 0 \}$.

Note: $GL(V)$ is an infinite group.

Let G be a finite group. We want to "represent" G by matrices.

DEF'N A representation of G is a group homomorphism $\varphi : G \to GL(V)$ for some v.s. V. In other words, for each $g \in G$ we have a matrix $\varphi(g)$, and:

- $\varphi(gh) = \varphi(g) \cdot \varphi(h) \; \forall \; g, h \in G$,
- $\varphi(e) = I_n$ identity matrix.

A representation of G is very similar to an action, except it is _linear_: we act by matrices, not permutations.

e.g For any V and any G, can set $\varphi(g)(v) = v$. $\forall v \in V$, i.e., $\varphi(g) = I_n$ identity matrix. This is called the _trivial representation_ and is boring...

e.g. Suppose $G \curvearrowright X$ a finite set. Let $\mathbb{C}[X] := \{ \sum_{x \in X} c_x x : c_x \in \mathbb{C} \}$ be v.s. of formal linear combinations of elements of X. Then $\mathbb{C}[X]$ is a G representation where $\varphi(g)(x) = g \cdot x$ for all basis vectors $x \in \mathbb{C}[X]$. In other words, each $\varphi(g)$ is the _permutation matrix_ of its corresponding permutation. This is called a _permutation representation_.

e.g. Let $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \cdots, n-1\}$. Let $V = \mathbb{C}$.
We can define a representation $\varphi: G \to GL(V)$ by
$\varphi(\kappa) = (e^{2\pi i \cdot k/n}) \leftarrow 1 \times 1$ matrix $\quad \forall \, k = 0, 1, \cdots, n-1$.

e.g. Let $G = S_n$ symmetric gp. and let $V = \mathbb{C}$ $\leftarrow 1 \times 1$ matrix.
The $\underline{\text{sign}}$ $\underline{\text{representation}}$ $\varphi: S_n \to GL(\mathbb{C})$ is $\varphi(\sigma) = (\bullet^{sgn(\sigma)})_{(-1)^{inv(\sigma)}} \leftarrow 1 \times 1$ matrix.

e.g. If U, V are G-representations, then, direct $\underline{\text{sum}}$ $U \oplus V$
is another representation: as matrices - $\left(\begin{array}{c|c} \varphi(g)_U & 0 \\ \hline 0 & \varphi(g)_V \end{array} \right) \leftarrow$ "block sum"

$\underline{\text{DEF'N}}$ A rep'n $\varphi: G \to GL(V)$ is $\underline{\text{irreducible}}$ if we
cannot find a $\underline{\text{nontrivial}}$ subspace U (i.e., $0 \neq U \neq V$)
s.t. $gu \in U \; \forall u \in U, g \in G$ (i.e., $\underline{\text{invariant}}$ under all G).

$\overset{\text{important}}{\hookrightarrow}$ $\underline{\text{FACT}}$ Every representation V of G is a direct sum
$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ of $\underline{\text{irreducible}}$ rep'ns V_i.

e.g. Let $V = \mathbb{C}^n$ w/ standard basis $\{e_1, e_2, \cdots, e_n\}$ and $G = S_n$,
Let $\varphi: S_n \to GL(V)$ be the standard permutation rep'n,
i.e. $\varphi(\sigma) e_i = e_{\sigma(i)} \; \forall \sigma \in S_n, i = 1, \cdots, n$. V is $\underline{\text{reducible}}$,
since $U_1 = \{(c, c, \cdots, c) \in V : c \in \mathbb{C}\}$ is a nontrivial invariant subspace.
With $U_0 = \{(x_1, \cdots, x_n) \in V : x_1 + \cdots + x_n = 0\}$, we have
$V = U_1 \oplus U_0$ and U_1, U_0 are $\underline{\text{irreducible}}$ rep'ns,
$\underset{\text{trivial rep'n}}{}$

The FACT above says that to understand all G-rep's,
it's enough to understand the irreducible ones...

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Characters of representations

Representations $\varphi: G \to GL(V)$ are matrix-valued functions, hence complicated to understand. It turns out we can "reduce" to studying "ordinary" $\mathbb{C}$-valued fn's $\chi: G \to \mathbb{C}$.

DEF'N Let φ be a representation of finite group G. Its __character__ $\chi_\varphi: G \to \mathbb{C}$ is the function
$$\chi_\varphi(g) = \text{Tr}(\varphi(g)) \leftarrow \underset{\text{matrix}}{\text{trace of}} \quad \text{for all } g \in G.$$

__e.g.__ If V is 1-dim'l, then φ and χ_φ are the same thing...

__e.g.__ If φ is the permutation rep'n of an action $G \curvearrowright X$ then $\chi_\varphi(g) = \# \text{Fix}(g: X \to X) \leftarrow$ why? think abt. perm. matrix...

FACT For two G-reps $\varphi_1: G \to GL(V_1)$, $\varphi_2: G \to GL(V_2)$ have $\chi_{\varphi_1} = \chi_{\varphi_2} \iff \varphi_1$ __isomorphic__ to φ_2

($\varphi_1 \cong \varphi_2$ means $\exists$ v.s. iso. $V_1 \cong V_2$ that commutes w/ G-action)

Upshot: enough to study characters, in fact, since we have $\chi_{\varphi_1 \oplus \varphi_2} = \chi_{\varphi_1} + \chi_{\varphi_2}$, enough to study characters of irreducible rep'ns (+ their lin. comb.'s).

In fact, Characters χ are not just any kind of function $G \to \mathbb{C}$...

DEF'N A __conjugacy class__ of G is set of the form $C = \{ghg^{-1} : g \in G\}$ for some $h \in G$. A function $f: G \to \mathbb{C}$ is called a __class function__ if it is constant on conjugacy classes, i.e. $f(h) = f(ghg^{-1}) \; \forall \, g, h \in G$. Let $Cl(G) := $ v.s. of class functions $f: G \to \mathbb{C}$.

Prop. Any character χ_φ is a class function.

Pf: $\chi_\varphi(ghg^{-1}) = Tr(ghg^{-1}) = Tr(g^{-1} \cdot gh) = Tr(h)$

        recall $Tr(AB) = Tr(BA)$ for matrices A, B ☒

FACT 1. $\{\chi_{\varphi_1}, \dots, \chi_{\varphi_K}\}$ is a basis of $Cl(G)$, where $\varphi_1, \dots, \varphi_K$ are all the irrep's of G (up to iso.).

     2. With the inner product $\langle \, , \, \rangle : Cl(G) \times Cl(G) \to \mathbb{C}$ given by $\langle f, f' \rangle := \frac{1}{\#G} \sum_{g \in G} f(g) \overline{f'(g)}$, the basis $\{\chi_{\varphi_1}, \dots \chi_{\varphi_K}\}$ is <u>orthonormal</u>.

     3. If $\varphi = \bigoplus_m c_m \varphi_m$ is decomposition of φ into irrep's, then $c_m = \langle \chi_\varphi, \chi_{\varphi_m} \rangle$.

<u>Note</u> in particular that
     #irreps (irreducible rep'ns) $= \dim Cl(G)$
                         $= $ #conjugacy classes of G.

e.g. G acts on itself by multiplication on the left, and corresponding perm. rep. is called the <u>regular rep'n</u> $\mathbb{C}[G]$. How does $\mathbb{C}[G]$ decompose into irrep's?

$\langle \chi_{\mathbb{C}[G]}, \chi_{\varphi_m} \rangle = \frac{1}{\#G} \sum_{g \in G} \underset{\underset{\#Fix(g: G \to G) = \begin{cases} \#G & \text{if } g=e \\ 0 & \text{otherwise} \end{cases}}{\|}}{\chi_{\mathbb{C}[G]}(g)} \overline{\chi_{\varphi_m}(g)}$

$= \frac{1}{\#G} \cdot \#G \cdot \overline{\chi_{\varphi_m}(e)} = \dim(\varphi_m).$

Hence

$\#G = \dim \mathbb{C}[G] = \dim\left(\bigoplus_m \dim(\varphi_m) \cdot \varphi_m\right) = \sum_m (\dim \varphi_m)^2$

                                               looks familiar...

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Characters of the Symmetric Group.

Finally, by focusing on case $G = S_n$, we see symmetric functions.

Prop. Two permutations $\sigma, \sigma' \in S_n$ belong to same conjugacy class $\iff$ they have the same cycle structure.

Pf: Exercise for you. ☒

So # conj. classes in S_n = # cycle structures = # partitions $\lambda \vdash n$

So # irrep's of S_n = # $\lambda \vdash n$, and in fact there is a standard way to index irrep's by partitions.

e.g. Let $\mathrm{triv}: S_n \to GL(\mathbb{C})$ be the trivial rep'n. Then
$$\mathrm{triv} = \ell_{\boxed{}}^{\ n\ boxes} = \ell_{(n)}$$

e.g. For $\mathrm{sgn}: S_n \to GL(\mathbb{C})$ sign rep'n, $\mathrm{sgn} = \ell_{\square} = \ell_{(1^n)}$.

e.g. Recall standard perm rep'n $\mathbb{C}^n = \underset{\mathrm{triv}}{U_1} \oplus \underset{\substack{\text{irreducible,} \\ \dim = n-1}}{U_0}$,
then $U_0 = \ell_{\boxed{}} = \ell_{(n-1,1)}$.

Write $X_\lambda = X_{\ell_\lambda}$ = character of irrep. indexed by $\lambda \vdash n$.

DEF'N The Frobenius characteristic $Fr: C\ell(S_n) \to \mathrm{Sym}(n)$

$\underset{\substack{\text{recall} = \\ \text{sym. fn's} \\ \text{of degree } n}}{}$

is given by $Fr(\delta_\lambda) = P_\lambda \leftarrow$ power sum

where δ_λ is class function $\delta_\lambda(\sigma) = \begin{cases} z_\lambda & \text{if cycle type}(\sigma) = \lambda \\ 0 & \text{otherwise} \end{cases}$

and $z_\lambda = \dfrac{n!}{1^{m_1} m_1! \, 2^{m_2} m_2! \cdots} = \dfrac{n!}{\#\text{ perm's in } S_n \text{ w/ cycle type}} = \lambda = (1^{m_1} 2^{m_2} \cdots)$.

Since the δ_λ are a basis of $C\ell(S_n)$ and P_λ are a basis of $\mathrm{Sym}(n)$, this is clearly a v.s. isomorphism.

Thm $Fr(\chi_\lambda) = S_\lambda$ ← Schur function.

This is (one reason) why Schur fn's are so important!

Cor $\dim \varphi_\lambda = f^\lambda = \#SYT$ of sh. λ

Pf: Via Fr, same as coeff. of $[x_1 x_2 \cdots x_n]$ in $S_\lambda = f^\lambda$ ✓ 📧

More generally ...

Cor If $\chi_\lambda(\mu) = $ ch. evaluated at a perm. of cycle type μ,
then $S_\lambda = \sum_\mu \chi_\lambda(\mu) \cdot z_\mu^{-1} P_\mu$.

∃ combinatorial rule for these coeff's, called the <u>Murnaghan-Nakayama</u> rule.

Also note that ... by the regular representation, have

$$n! = \#S_n = \sum_{\lambda \vdash n} \dim(\varphi_\lambda)^2 = \sum_{\lambda \vdash n} (f^\lambda)^2, ~~~~~~~~~~$$

which we saw earlier using R.S. algorithm.

Finally, using something called the <u>induction product</u> of representations of $S_k \times S_{n-k} \longrightarrow S_n$,
we can get ring <u>structure</u> on $Sym = \bigoplus Sym(n)$.

Structure constants $S_\lambda \cdot S_\mu = \sum_\nu c_{\lambda \mu}^\nu S_\nu$ are called <u>Littlewood-Richardson</u> coefficients, also very important!

e.g. Character table for S_3:

	$(1)(2)(3)$	$(12)(3),$ $(13)(2),$ $(1)(23)$	$(123),$ (132)
$\chi_{triv} = \chi_{\boxed{\square\square\square}}$	1	1	1
$\chi_{sgn} = \chi_{\boxed{\square}}$	1	-1	1
$\chi_{std} = \chi_{\boxed{\square}}$	2	0	-1

to compute these χ_i
use $S_3 \circlearrowright \mathbb{C}^3 = \rho_{triv}^\wedge \oplus \rho_{std}^\wedge$ "$\mathcal{U}_0$"

So $\chi_{std}(\sigma) = \#Fix(\sigma)$
-1 ✓

So e.g. $S_{(2,1)} = \frac{1}{3!} \left(2 \cdot 1 \cdot P_{(1,1,1)} + 0 \cdot 3 \cdot P_{(2,1)} + -1 \cdot 2 \cdot P_{(3)} \right)$ //

Thanks for taking my course!

There are many more things to
be said about symmetric functions
(+ combinatorics in general)
So please don't hesitate to ask me about
anything you might be interested in
learning more about.

Have a nice summer!