Howard Math 274, HW# 1,

Spring 2022; Instructor: Sam Hopkins; Due: ~~Friday, February 11th~~ Monday, February 14th

1. A *k-ary necklace of length n* is a rotation equivalence class of colorings of the vertices of an n-gon with k colors. Use unweighted Pólya counting to show that the number of k-ary necklaces of length n is
$$\frac{1}{n}\sum_{d|n}\varphi(d)k^{\frac{n}{d}}.$$
This formula uses some notation from number theory: $d \mid n$ means "d divides n"; and $\varphi(d)$ is *Euler's totient function*, the number of $1 \leq j \leq d$ with $\gcd(d, j) = 1$.

2. Continuing the previous problem, now using weighted Pólya counting: how many ways, up to rotation, can the vertices of a hexagon be colored with 2 red, 2 green, and 2 blue vertices?

3. There are 24 orientation-preserving symmetries of a cube– they are all spatial rotations. Use unweighted Pólya counting to give a formula for the number of ways, up to orientation-preserving symmetries, to color the faces of a cube with k colors.

   Hint 1: Your formula should be a polynomial in k.

   Hint 2: This group of symmetries is *abstractly* isomorphic to the symmetric group S_4 (but of course there are *six*, not four, faces of a cube); for more information on this group see for instance the Wikipedia page `https://en.wikipedia.org/wiki/Octahedral_symmetry`.

4. Continuing the previous problem, now using weighted Pólya counting: how many ways, up to orientation-preserving symmetries, can the faces of a cube be colored with 2 red, 2 green, and 2 blue faces?

5. Let $\mathcal{M}_{n\times m}(k)$ be the set of $n\times m$ matrices with entries from the set $\{1, 2, \ldots, k\}$. For example,
$$\begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 1 & 2 & 3 \\ 4 & 3 & 5 & 2 \end{pmatrix} \in \mathcal{M}_{3\times 4}(5).$$
The symmetric group S_n acts on $\mathcal{M}_{n\times m}(k)$ by permuting rows: e.g., for $\sigma = (1, 2)(3) \in S_3$,
$$\sigma \cdot \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 1 & 2 & 3 \\ 4 & 3 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 3 \\ 2 & 3 & 4 & 2 \\ 4 & 3 & 5 & 2 \end{pmatrix}.$$
Let $\widetilde{\mathcal{M}}_{n\times m}(k)$ denote the set of S_n-equivalence classes of $\mathcal{M}_{n\times m}(k)$. Give a formula (in terms of n, m, and k) for $\#\widetilde{\mathcal{M}}_{n\times m}(k)$.

   Hint: To simplify your formula you may use the fact, which we proved last semester, that the (unsigned) Stirling numbers of the 1st kind $c(n, j) := \#\{\sigma \in S_n : \sigma \text{ has } j \text{ cycles}\}$ have generating function $\sum_{j=1}^{n} c(n, j)t^j = t(t + 1)\cdots(t + n - 1)$.

   Hard bonus problem, just to think about: What if I'm allowed to independently permute both *rows and columns* of the matrix?