

Math 4707: Introduction to Combinatorics and Graph Theory

Lecture Addendum, October 4, 2010

Recurrence Relations and a Glimpse of Generating Functions

Given an infinite sequence of numbers, a *generating function* is a compact way of expressing this data.

1 Example 1: The Binomial Theorem

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n = (x + 1)^n.$$

The point is that if n is really large, the sum on the left-hand-side can be quite long, however the right-hand side is always a relatively compact and short expression. We call the right-hand-side the *generating function* for the sequence of numbers $\left\{1, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots\right\}$.

Notice, in this case, this sequence of numbers eventually terminates. This is because $\binom{n}{n+1}, \binom{n}{n+2}, \dots = 0$. However, even if this sequence of numbers was infinite, we can talk about its generating function. We will see an example of this in the next section. The general form is that a sequence of numbers

$$\{c_0, c_1, c_2, c_3, \dots\}$$

has the associated generating function

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

Two very common uses of Generating Functions are the following:

1) Sometimes, it is easier to find the generating function for a sequence of numbers than a formula that works for all of the numbers themselves. However, with the generating function in hand, one can find any given coefficient more easily, either using Taylor Expansions or other techniques.

2) Sometimes, it is possible to deduce properties of a sequence of numbers from the generating function more easily than from the numbers themselves. We already saw an example of this with the binomial coefficients:

The algebraic statement $(x + 1)^n(x + 1)^n = (x + 1)^{2n}$ could be used to deduce the summation identity:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \dots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

2 A more complicated example

We have seen $\binom{-n}{k} := \frac{-n(-n-1)\dots(-n-k+1)}{k!}$ also equals $(-1)^k \binom{n+k-1}{k}$, which equals $(-1)^k$ (# multisets of size k from an n -set).

Let us consider the sequence $\binom{-3}{k}$ for $k \geq 0$:

$$1, -3, 6, -10, 15, -21, 28, \dots$$

What is the generating function for this sequence?

In fact, the associated generating function is $\frac{1}{(x+1)^3} = (x+1)^{-3}$. Let us recall Taylor's theorem from calculus. If we expand $f(x)$ as a series about $x = 0$, then we obtain

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

If $f(x) = (x+1)^{-3}$, we obtain $f(0) = 1$, $f'(x) = (-3)(x+1)^{-4}$ so $f'(0) = -3$. $f''(x)$ is the derivative of $(-3)(x+1)^{-4}$ and so we obtain $f''(x) = (-3)(-4)(x+1)^{-5}$ and so $\frac{f''(0)}{2!} = 12/2 = 6$. Continuing this way, we see that the coefficient of x^k in this expansion is $\binom{-3}{k}$. In particular, one can use similar tools to prove the *Generalized Binomial Theorem*:

(Generalized Binomial Theorem) If n is any real number (e.g. including negative integers) then the generating function for the sequence

$$1, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots$$

is the function $(1+x)^n$.

Exercise: Think about the sequence $1, \binom{-1}{1}, \binom{-1}{2}, \binom{-1}{3}, \dots$. Compare the generating function you get with the expression for the *geometric series* $1 + x + x^2 + x^3 + x^4 + \dots$.

3 Fibonacci Numbers

Recall that the Fibonacci numbers are defined by the properties $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and $F_1 = 1, F_2 = 1$. We wish to find the generating function for the sequence $\{0, F_1, F_2, F_3, \dots\}$. In other words, we want to write the series

$$0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + 21x^8 + 34x^9 + 55x^{10} + 89x^{11} + 144x^{12} + 233x^{13} + \dots$$

as a small compact expression.

Let us assume that $F(x)$ denotes the generating function. Since we know that $F_n = F_{n-1} + F_{n-2}$ we know that for n greater than or equal to 2, the coefficient of x^n in $F(x)$ is equal to the sum of the $(n-1)$ th coefficient and the $(n-2)$ nd coefficient.

We thus get the expression $x + xF(x) + x^2F(x) = F(x)$ from the equation

$$\begin{aligned} &0 + 1x \\ &+ 0x \quad + \quad 1x^2 + 1x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} + \dots \\ &\quad + \quad 0x^2 + 1x^3 + 1x^4 + 2x^5 + 3x^6 + 5x^7 + 8x^8 + 13x^9 + 21x^{10} + \dots \\ = &0 + 1x \quad + \quad 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + 21x^8 + 34x^9 + 55x^{10} + \dots \end{aligned}$$

We can then solve for $F(x)$: $F(x)(1 - x - x^2) = x$. Thus

$$F(x) = \frac{x}{1 - x - x^2}.$$

If we look at the Taylor series for this rational function, we indeed obtain coefficients that are the Fibonacci numbers. Generating Functions are also helpful for obtaining closed formulas or asymptotic formulas.

If we use partial fraction decomposition, we see that

$$F(x) = \frac{A}{1 - \lambda_1 x} + \frac{B}{1 - \lambda_2 x}.$$

In particular, the right-hand-side sums up to be

$$\frac{A(1 - \lambda_2 x) + B(1 - \lambda_1 x)}{(1 - \lambda_1 x)(1 - \lambda_2 x)} = \frac{(A + B) - (B\lambda_1 + A\lambda_2)x}{1 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 x^2}.$$

Comparing this fraction with the above expression for $F(x)$, if we solve for λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1\lambda_2 = -1$, we are exactly solving for the two roots of the equation $x^2 + x - 1 = 0$. These are exactly the *golden ratio* and its inverse

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2}.$$

Solving for A and B so that $(A + B) = 0$ and $A\lambda_2 + B\lambda_1 = 1$ we obtain

$$F(x) = \frac{1/\sqrt{5}}{1 - x(1 + \sqrt{5})/2} - \frac{1/\sqrt{5}}{1 - x(1 - \sqrt{5})/2}.$$

These expressions look complicated, so let us use the notation $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ to make the next logic clearer. Let us also let $c = \frac{1}{\sqrt{5}}$. Then we obtain

$$F(x) = \frac{c}{1-\phi x} - \frac{c}{1-\bar{\phi}x} = \left(c + c\phi x + c\phi^2 x^2 + c\phi^3 x^3 + \dots \right) + \left(-c - c\bar{\phi}x - c\bar{\phi}^2 x^2 - c\bar{\phi}^3 x^3 - \dots \right).$$

Thus we conclude the following closed formula for the Fibonacci numbers:

(Formula for the n th Fibonacci number) $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$

Exercise: Check this formula for $n = 1, 2, 3, 4, 5, 6$.

4 Asymptotic Formula for F_{n+1}/F_n

We can use the closed formula to obtain an expression for F_{n+1}/F_n as n gets large.

Notice that $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\dots$ and $\frac{1-\sqrt{5}}{2} \approx -0.618\dots = -(1-\phi)$. Consequently, as $n \rightarrow \infty$, $\left(\frac{1-\sqrt{5}}{2} \right)^n = (-0.618)^n \rightarrow 0$. Thus

$$F_n \rightarrow \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - 0$$

as $n \rightarrow \infty$.

Thus $F_{n+1}/F_n \rightarrow \left(\frac{1+\sqrt{5}}{2} \right)$, the golden ratio 1.618.... There are examples of this on page 72 of LPV.