Math 4707: Intro to combinatorics and graph theory

1/20/21

Ch. 1

LVP

Plan for today:

- Go over course logistics
- Do introductions :)
- Overview of course
- Chapter 1: Basic Counting
- Maybe... try some groupwork

Logistics

- Instructor: Sam Hopkins (Me!). Call me Sam.
  email: shopkins@umn.edu
- Class time: Mon. Wed. 2:30-4:25 pm on Zoom
- Office hrs: online, by appointment ← can change if wanted
- textbook: "Discrete Mathematics" by Lovász et al.
- Assessments (all 'take-home'):
  5 HW's, 2 Midterms, 1 Final
  collaboration encouraged — no collaboration
- Course website:
  math.umn.edu/~shopkins/classes/4707.html

Introductions:

- Say who you are (how you'd like to be called, pronouns if you want, etc.)
- Say where you are (we're all over!)
- Say one thing you've been doing to stay grounded during quarantine

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Overview of course

This is a course in discrete math

Discrete

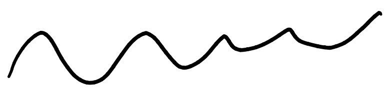
finite

integers $\mathbb{Z}$

algebra (ish)

computer science

Continuous



infinite

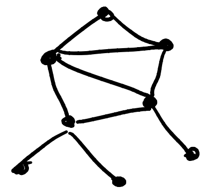
real numbers $\mathbb{R}$

calculus

(classical) Physics

There will be 3 major topics we cover:

- enumerative combinatorics
  = counting discrete structures
- graph theory
  = study of 'networks' like



- optimization (+ algorithms)
  = finding the 'best' discrete structure

We will cover these topics in roughly this order, but there will be alot of overlap and call backs, etc.

Some other similar courses UMN offers:

- Math 5705: Enumerative Combinatorics (last semester)
- Math 5707: Graph theory (this semester)

(5 min. break before math?)

Basic counting

For the first several weeks we'll talk abt. counting.
In Ch. 1 of the book they introduce several basic counting problems via 'realworld' scenarios...

Here's an example:

Seven people meet at a party. Each person shakes hands wit each other person. How many total handshakes occur?

ANSWER: 21

Two possible solutions are:

#1 Each person shakes hands w/ each other person, so $7 \times (7-1) = 7 \times 6 = 42$ hands shake. But a handshake involves two hands shaking, so we have to divide by 2 to get $42/2 = 21$ handshakes. ☒

#2 Imagine the 1st person shakes hands w/ each other person, then the 2nd person shakes hands w/ each other person except the 1st, then the 3rd shakes everyone except #'s 1, 2, + 3, et cetera. This way we will count each handshake once.
The 1st person does $(7-1)=6$ handshakes, then the 2nd person does $(7-2)=5$, 3rd person does 4, etc.

Total # of handshakes $= 6+5+4+3+2+1+0 = 42$. ☒

Aside, Recall: $1+2+3+\dots+(n-1) = \dfrac{n(n-1)}{2}$. $(\ast)$

Can prove this using (mathematical) induction.

Setting for induction: Have a statement $P(n)$ depending on a parameter (= number) n.

If you can:

- (base case) show the case $P(1)$ holds.
- (induction step) show that $P(n)$ implies $P(n+1)$

then you've proved $P(n)$ for all $n = 1, 2, \dots$!

e.g. $(\ast)$ $n=1 \rightsquigarrow 0 = \frac{1\cdot 0}{2} = 0$ ✓

(induction step) $1+2+\dots+(n-1)+n = \dfrac{n(n-1)}{2} + n = \dfrac{n^2-n+2n}{2} = \dfrac{n(n+1)}{2}$ ✓

The book discusses several other counting word problems, but let's move on right away to a more formal mathematical framework for counting.

Sets A **set** is any collection of objects. The objects are called the **elements** of the set.

In math, we often deal w/ sets of numbers, like $\{1,2,3,4\}$ or $\{\ldots,-2,-1,0,1,2,\ldots\} = \underset{\text{integers}}{\mathbb{Z}}$

But sets can be made of any kind of objects.
{Earth, Mars, Venus} is a set of planets.
{Alice, Bob, Carol, David, Eve, Frank, George} is a set of party-goers.

You can see we use { + } ; (braces) to show sets.

We also use "set-builder notation":

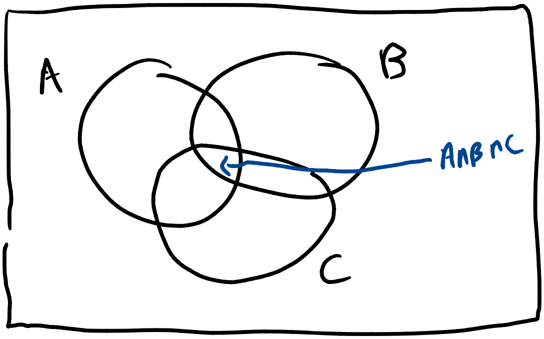
$$\{x \overset{\text{"in"}}{\in} \mathbb{Z} \underset{\text{"such that"}}{:} \underset{\text{condition}}{x \geq 0}\} = \{0,1,2,\ldots\} = \mathbb{N}$$

nonnegative integers
"natural numbers"

Important **operations** on sets include
- $\cap$ = **intersection** ('and')
- $\cup$ = **union** ('or')

e.g. $\{-4, 1, 6\} \cap \mathbb{N} = \{1, 6\}$.

You may be used to representing intersections and unions of sets using Venn diagrams:



We'll talk more abt Venn diagrams soon!

For right now, the most important set concept for us will be subset. A subset of a set A is any sub-collection of the elements of A.

e.g. $\{\text{Mars, Venus}\} \subseteq \{\text{Earth, Mars, Venus}\}$

is subset of

Note: The empty set $\phi = \{\ \}$, which has no elements, is a subset of every set.

Q: How can we reformulate the handshake problem in terms of sets + subsets?

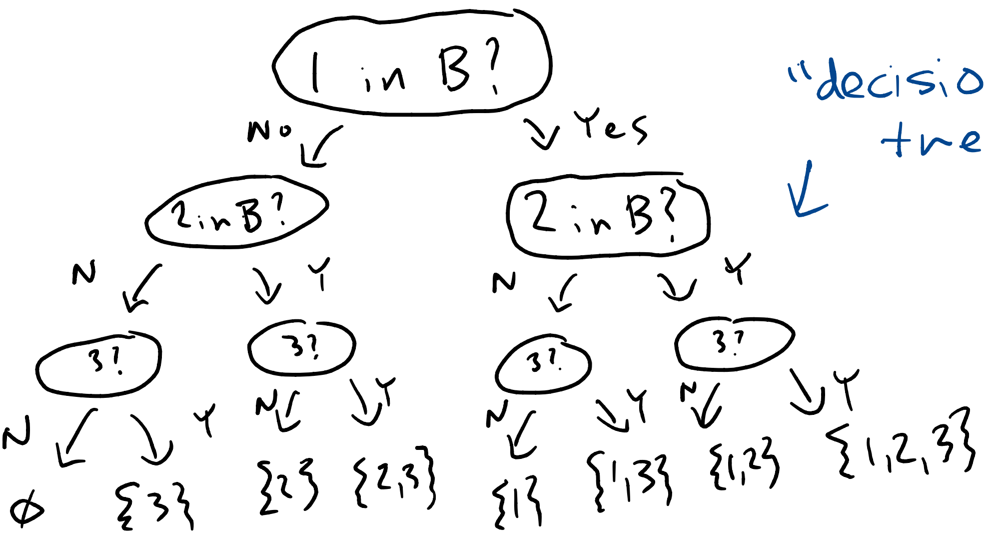
Counting all subsets

Let $A = \{1, 2, \ldots, n\}$ be an n-element set. How many subsets of A are there?

ANSWER: 2^n

Pf: Consider making a subset B of A by considering each of $1, 2, 3, \ldots, n$ in turn and deciding whether or not to include that number in your subset. E.g.,

$\boxed{n=3}$



"decision tree"

Have n independent choices, each of 2 possibilities $= 2 \times 2 \times \cdots \times 2 = 2^n$ total options $\blacksquare$

Sequences A sequence (or word) is a list $a_1 a_2 \cdots a_k$ of numbers in order. The a_i are called the letters, and if all a_i belong to set A, then A is the alphabet.

e.g. 11435 is a sequence of length 5 from alphabet $\{1,2,3,4,5\}$

note: 2 doesn't have to appear!

How many sequences of length k from alphabet $\{1,2,\ldots,n\}$ are there?

ANSWER: n^k

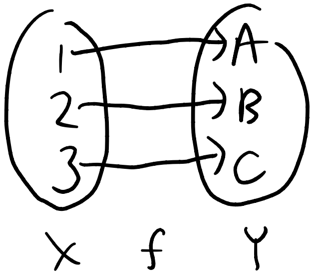
Pf: Similar to subsets. For seq. $a_1 a_2 \cdots a_k$, have k independent choices of the a_i's, and each a_i can be one of n things

$\Rightarrow$ $n \times n \times \cdots \times n = n^k$ total options. ☑

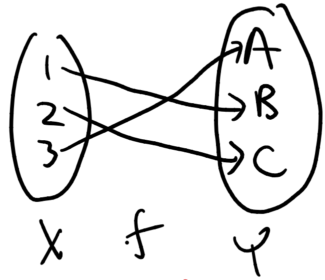
Aside,

Bijections A bijection $f: X \to Y$ is a one-to-one correspondence that matches each $x \in X$ to a unique $y \in Y$ and vice-versa.

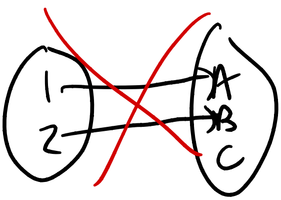
e.g.

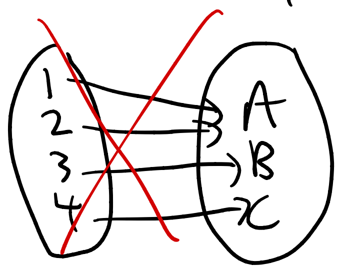


or



but NOT





Bijections are useful for counting because if there is a bijection $f: X \to Y$ then

number of elements of $X \to$ $$\#X = \#Y$$

E.g. Can you describe a bijection

$$f: \{\text{subsets of } \{1,2,\dots,n\}\} \to \left\{\begin{array}{c}\text{length } n \\ \text{sequences w/} \\ \text{alphabet } \{1,2\}\end{array}\right\}?$$

HINT: think about our above proofs ...

Permutations A permutation of $\{1, 2, \cdots, n\}$ is a sequence $a_1 a_2 \cdots a_n$ where each number in $\{1, 2, \ldots, n\}$ appears exactly once.

e.g. 53142 and 15234 are perm.'s of length 5.

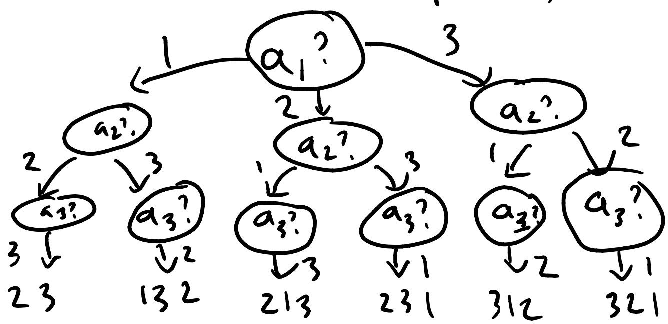
How many perm.'s of length n are there?

ANSWER: $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

'factorial'

Pf: Consider choosing the letters of our perm. one at a time. For the 1^{st} letter we can choose any of n numbers; for 2^{nd} letter we have $(n-1)$ choices b/c can't repeat, etc.

$\boxed{n=3}$



Total options $= n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$ ▨

Ordered subsets An **ordered subset** of $\{1,2,\dots,n\}$ is a sequence $a_1 a_2 \cdots a_k$ where each number in $\{1,\dots,n\}$ appears at most once.

e.g. $1\,3\,4$ is an ordered subset of $\{1,2,3,4,5\}$, and $4\,1\,3$ is a different one.

How many ordered subsets of $\{1,2,\dots,n\}$ of size k are there?

ANSWER: $n \times (n-1) \times (n-2) \times \cdots \times (n-(k-1))$.

Pf: Same as in permutations case, consider selecting letters one-by-one. Have n choices for 1^{st} letter, $(n-1)$ for 2^{nd}, all the way down to $(n-(k-1))$ for k^{th}. $\blacksquare$

This brings us to probably the most important basic counting problem...

Subsets of given size

How many subsets of $\{1, \ldots, n\}$ of size k are there?

ANSWER: $\dfrac{n!}{k!(n-k)!}$

Proof: We saw that the # of ordered subsets

$$= n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}.$$

But for each (unordered) k-subset, there are $k!$ ways to order it:

e.g., $\{1,3,4\} \longrightarrow$ 134, 314, 413, 143, 341, 431

So we have to divide by $k!$ $\Rightarrow$ $\dfrac{n!}{k!(n-k)!}$ ▨

These numbers are so important, they have a special notation + name:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

"n choose k"

'binomial coefficients'

e.g. Returning to handshake problem:

$$\left\{\begin{array}{l}\text{handshakes} \\ \text{among 7 people}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{size 2} \\ \text{subsets} \\ \text{of } \{1, \cdots 7\}\end{array}\right\} \Rightarrow \binom{7}{2} = \frac{7 \cdot 6}{2} = 21$$

handshakes.

e.g. How many 5 card poker hands from a standard 52 card deck are there?

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \sim 2.5 \text{ million}$$

If there's any time left... we can go into breakout groups and start a worksheet where we find the probabilities of the different poker hands.

We'll finish this worksheet next class.

NOTE: $$\text{Prob.}\left(\begin{array}{c}\text{certain} \\ \text{kind of} \\ \text{hand}\end{array}\right) = \frac{\#\text{ of that kind of hand}}{\text{total }\#\text{ of hands}}$$