Math 4707: Graphs + Linear Algebra

3/1

Not in LPV

Reminders: • Midterm #1 has been graded.

- HW #2 should be graded soon...
- HW#3 has been posted, is due Wed. 3/10.

Today we will talk about how matrices and linear algebra can help us understand things about graphs.

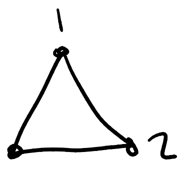
Recall from last class the adjacency matrix $A_G = (a_{ij})$ of a graph G given by

$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \text{ an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

for multigraphs, a_{ij} = # edges between i + j

e.g.

$G =$



$$A_G = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array}$$

Notice that A_G is always symmetric, i.e. $a_{ij} = a_{ji}$.

So far it seems like A_G is just a re-encoding of G (and not an efficient one). To understand importance of A_G, need to think abt things we can do with matrices, like multiplying them.

Counting walks

Remember that a walk from vertex u to vertex v of length ℓ is a sequence of vertices/edges:

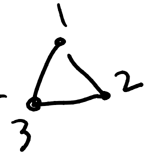
$$u = v_0, e_1, v_1, e_2, \cdots, v_{\ell-1}, e_\ell, v_\ell = v$$

w/ $e_i = \{v_{i-1}, v_i\}$.

Thm The (i,j) entry of A_G^ℓ is the # of walks of length ℓ from i to j.

Gives a direct graph theory interpretation of matrix multiplication!

e.g.

$G =$ 

$A_G = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$ $A_G^2 = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix} = \begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$

walks len. 2 $1 \to 1 = 2$ (1-2-1, 1-3-1)

walks len. 2 $1 \to 2 = 1$ (1-3-2)

Pf: By induction on ℓ. Base case $\ell = 0$:

$$A_G^0 = I = \begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}.$$

$$\#\text{walks len. } 0\ i \to j = \begin{cases}1 & i=j\\ 0 & \text{otherwise}\end{cases}$$ ✓

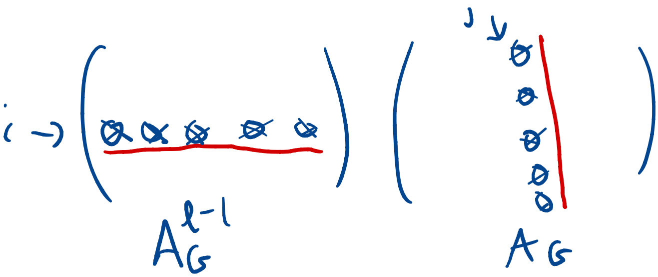
Now suppose result holds for $\ell - 1$. Lets use $M[i,j]$ to denote (i,j) entry of M.

Clear that

$$\#\text{walks len. } \ell\ i \to j = \sum_{\text{vert's } k} (\#\text{walks len. } \ell-1\ i \to k) \cdot (\#\text{walks len } 1\ k \to j)$$

$$= \sum_k \overset{\text{induction}}{A_G^{\ell-1}[i,k]} \cdot \overset{\#\text{edges } k\to j}{A_G[k,j]}$$

$$= A_G^{\ell}[i,j]$$ by def. of matrix mult.



✓

□

There's a special choice of i, j w/ especially nice answer. Remember that the trace of M is

$$\operatorname{tr}(M) = \sum_{i=1}^{n} M[i,i]$$

(sum of diagonal entries)

Cor The total # of closed walks in G (starting at any vertex) of length ℓ is $= \operatorname{tr}(A_G^{\ell})$.

Can give an even better formula w/ eigenvalues.

Basic result from lin. algebra says that because A_G is real and symmetric it is diagonalizable, i.e., ∃ invertible matrix P s.t.

$$A_G = P \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} P^{-1}, \text{ where}$$

$\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A_G (w/ mult.).

Cor # closed walks of len. ℓ in G =

$$\lambda_1^{\ell} + \lambda_2^{\ell} + \cdots + \lambda_n^{\ell}$$

Pf: # closed walks length ℓ

$\swarrow$ ℓ copies

$$= \operatorname{tr}(A_G^{\ell}) = \operatorname{tr}\left(P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \underline{P^{-1}} \cdot \underline{P} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1} \cdots \right)$$

$$= \operatorname{tr}\left(P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^{\ell} P^{-1} \right)$$

tr(AB) = tr(BA)

$$= \operatorname{tr}\left(P \begin{pmatrix} \lambda_1^{\ell} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{\ell} \end{pmatrix} P^{-1} \right) = \operatorname{tr}\begin{pmatrix} \lambda_1^{\ell} & & & \\ & \lambda_2^{\ell} & & \\ & & \ddots & \\ & & & \lambda_n^{\ell} \end{pmatrix} =$$

$$\lambda_1^{\ell} + \lambda_2^{\ell} + \cdots + \lambda_n^{\ell} \quad ✓$$

▨

e.g.

$G =$ 

$$A_G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

How to find eigenvalues of A_G?

The roots of the characteristic polynomial!

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

So eigenvalues are 1, -1.

$$\Rightarrow \#\text{ closed walks len. } \ell = 1^{\ell} + (-1)^{\ell} = \begin{cases} 2 & \ell \text{ even,} \\ 0 & \ell \text{ odd.} \end{cases}$$

Closed walks $1-2-1-\cdots-1$

$2-1-2-\cdots-2$ $\leftarrow$ only even length!

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Counting spanning trees

We just saw a very cool graph-theoretic interpretation of the trace of a matrix. What's the other most important number associated to a matrix?

Determinant!

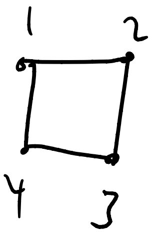
The Laplacian matrix L_G of G is

$$L_G = \begin{pmatrix} \deg(v_1) & & & 0 \\ & \deg(v_2) & & \\ & & \ddots & \\ 0 & & & \deg(v_n) \end{pmatrix} - A_G .$$

The reduced Laplacian matrix $\tilde{L}_G$ is obtained from L_G by deleting the i^{th} row and i^{th} column, for any choice of i.

e.g.

$G =$



$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

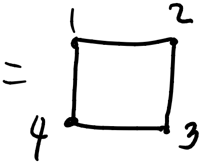
$$L_G = \begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 2 \end{pmatrix} - A_G = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\tilde{L}_G = \begin{pmatrix} 2 & -1 & 0 & \cancel{-1} \\ -1 & 2 & -1 & \cancel{0} \\ 0 & -1 & 2 & \cancel{-1} \\ \cancel{-1} & \cancel{0} & \cancel{-1} & \cancel{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Thm (Kirchoff's Matrix-Tree Thm)

$$\#\text{ spanning trees of } G = \det(\tilde{L}_G)$$

e.g. $G =$

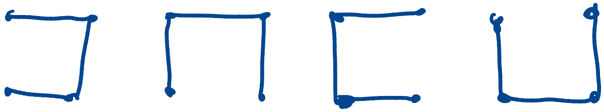


$$\tilde{L}_G = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\det(\tilde{L}_G) = \begin{array}{l} (2\cdot2\cdot2 + -1\cdot-1\cdot0 + 0\cdot-1\cdot-1) \\ -(2\cdot-1\cdot-1 + -1\cdot-1\cdot2 + 0\cdot2\cdot0) \end{array}$$

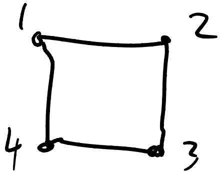
$$= 8 - 2 - 2 = 4$$

spanning trees:

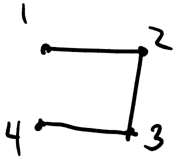


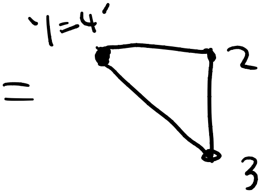
Won't give the whole proof, but let's sketch the idea, again based on induction.

For edge $e = \{u,v\}$ of G, the deletion of e, denoted $G-e$, is graph we get by removing e, while the contraction of e, denoted G/e is graph obtained by 'squishing' vertices u and v together:

e.g. $G =$ 

$e =$ 

$G - e =$ 

$G/e =$ 

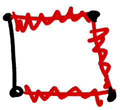
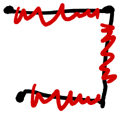
Lemma # spanning trees of G

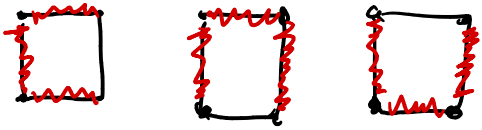
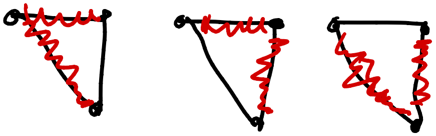
$=$ # spanning trees of $G-e$ + # sp. trees of G/e

Pf: $\exists$ simple bijections

spanning trees of G that don't use e ⟷ spanning trees of $G-e$

spanning trees of G that do use e ⟷ spanning trees of G/e

e.g.  ⟷ 

 ⟷ 

↖ think abt. this

∎

To prove Matrix-Tree Thm, need to show $\det(\tilde{L}_G)$ satisfies same recurrence. Can do this w/ basic rules for det — see note of G. Musiker.

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With Matrix-Tree Thm we can give another proof of Cayley's formula for the # of labeled trees on n vertices:

$$\#\text{ trees on } [n] = \#\text{ spanning trees complete graph } K_n$$

$$\underset{\text{matrix-tree thm}}{=} \det(\tilde{L}_{K_n})$$

$(n-1)\times(n-1)$ matrix

$$= \det\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \ddots & \vdots \\ -1 & \cdots & \cdots & \cdots & n-1 \end{pmatrix}$$

$$\underset{\text{add every other row to 1st row}}{=} \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ -1 & \cdots & & \cdots & n-1 \end{pmatrix}$$

$$\underset{\text{add 1st row to every other row}}{=} \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & 0 \cdots & 0 \\ 0 & 0 & n \cdots & 0 \\ 0 & 0 & \cdots & n \end{pmatrix} \underset{\substack{\uparrow \\ \text{upper triang!}}}{=} n^{n-2} \quad \checkmark$$

Now let's take a 5 min. break and when we come back we can practice w/ matrices + graphs on today's worksheet in breakout groups!