Math 4707: Matchings + the Marriage Thm.

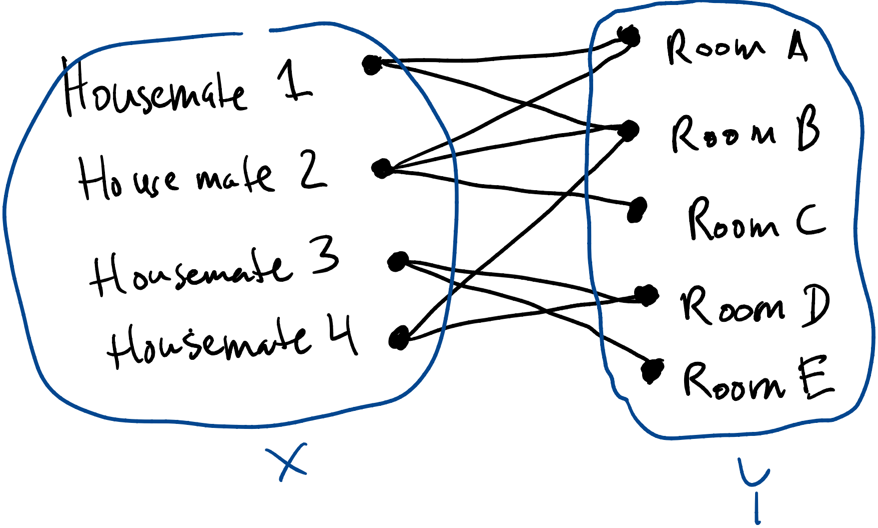
3/10

Ch. 10 of LPV

Reminder: • HW #3 is due today.

Consider the following scenario: a group of people decide to be housemates; they find a house that has a number of different rooms in it; and they want to know if there's a way of assigning (unique) rooms to each person s.t. everyone is given a room they find acceptable.

How might we model this problem? The information of which rooms housemates find acceptable is naturally encoded in a special kind of graph. Namely, consider the graph G whose vertex set $V = X \cup Y$ consists of two kinds of vertices: the set $X = \{\text{housemates}\}$ and the set $Y = \{\text{rooms}\}$; and we draw an edge from $x \in X$ to $y \in Y$ if housemate x finds room y acceptable:



Notice what's special about this graph is that its vertices are partitioned into two parts, X and Y, s.t. edges only exist between vertices in different parts (no edges within X or within Y).

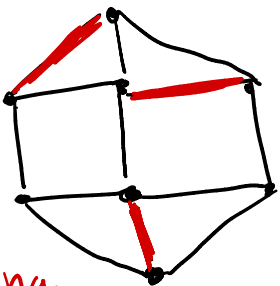
We call a graph like this a bipartite graph, with (X, Y) being its bipartition.

A solution to the housemates problem (i.e., a way of assigning rooms to housemates) is a certain substructure in this bipartite graph: it is an "(X-saturating) matching." So we will now define and study matchings...

Def'n Let G be a graph. A *matching* M in G is a subgraph of G consisting of vertex-disjoint edges. M is a *perfect matching* if every vertex of G belongs to some edge of M.

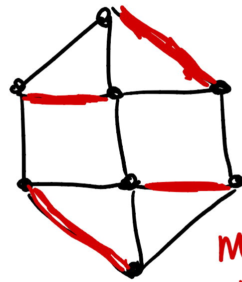
e.g.

$G =$



M matching

$G =$



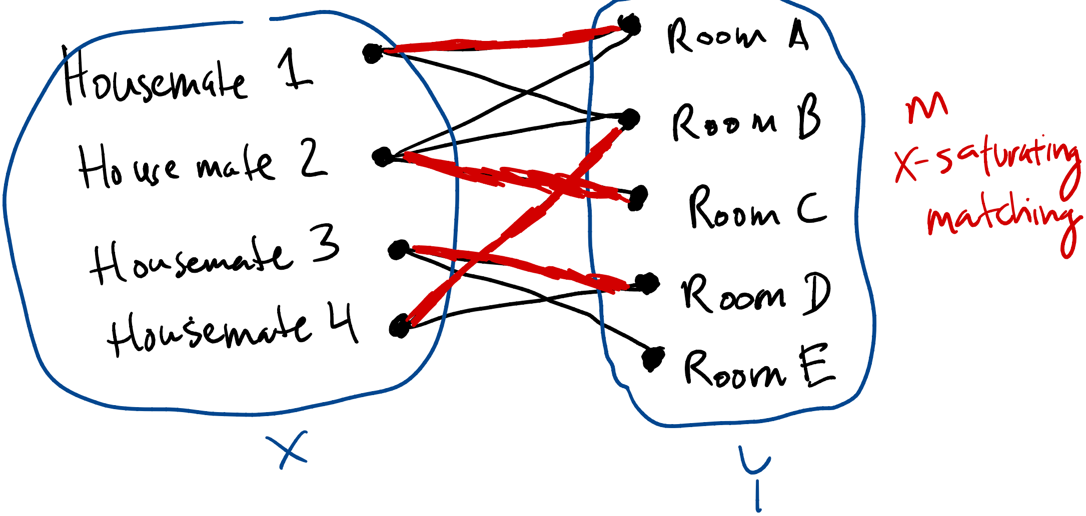
M perfect matching

The notion of matching makes sense for any graph, but we will only study matchings in bipartite graphs. We also want to capture idea that all housemates should get a room ...

Def'n Let G be a bipartite graph w/ bipartition (X, Y). An *X-saturating matching* M is a matching containg all vertices in X.

Note: If $\#X = \#Y$, then matching M is X-saturating $\Leftrightarrow$ M is perfect.

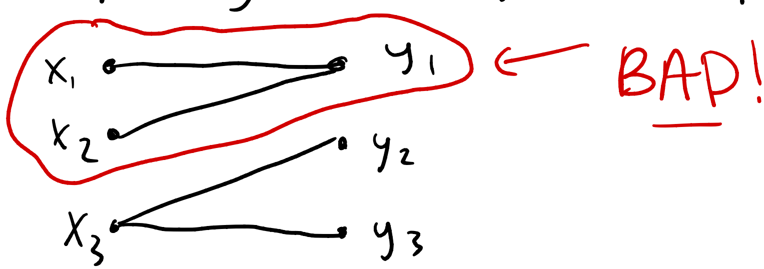
e.g.



We now see that a sol'n to the housemates problem = an X-saturing matching in G.

So we want to figure out when these matchings exist. Let's first think of necessary conditions...

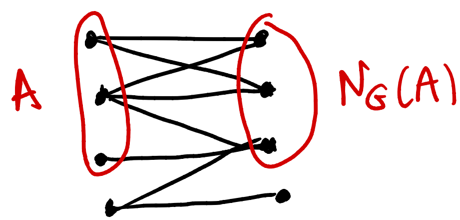
One obvious necessary condition: every $x \in X$ has to be adjacent to at least one $y \in Y$ (i.e., no "isolated vertices" in X). Similarly, for any two vertices $x_1, x_2 \in X$, there better be at least two vertices in Y among vertices adjacent to x_1 or x_2:



Indeed, if x_1, x_2 only have one vertex adjacent to either, then Pigeonhole Principle says there is no X-saturating matching. For more necessary conditions, we can consider triples of vertices in X, and so on.... Motivates following definition:

Def'n Let G be a graph and $A \subseteq V$ a subset of its vertices. The neighborhood of A, denoted $N_G(A)$, is the set of all vertices $v \in V$ that are adjacent to at least one vertex in A.

e.g.



The necessary conditions we get from the pigeonhole principle become:

Prop: Let G be bipartite w/ (X, Y) bipartition. If there is any subset $A \subseteq X$ for which

$$\# N_G(A) < \# A,$$

then G does not have an X-saturating matching.

Note: Taking $A = X$, we see that $\# Y \geq \# X$ is a requirement (need as many rooms as people).

The surprising fact is that these necessary conditions are sufficient:

Thm (Hall's Marriage Theorem) G as above. Then an X-saturating matching in G exists $\Leftrightarrow$ for all $A \subseteq X$, $\# N_G(A) \geq \# A$.

~ Often this is stated just for perfect matchings:

Thm G as above, w/ $\# X = \# Y$. Then there is a perfect matching in $G \Leftrightarrow$ for all $A \subseteq X$, $\# N_G(A) \geq \# A$.

Note: Name "marriage theorem" comes from viewing X = men, Y = women, and matching M = way of marrying all men to all women. I think the housemates story is less outdated...

Even if we cannot find an X-saturing matching in our bipartite graph G, we might still want to find the biggest matching we can. There's an extension of marriage thm. that answers this as well!

Thm G bipartite w/ bipartition (X,Y). Then maximum size of matching in G =

$$\#X - \max_{A \subseteq X} \left(\#A - \#N_G(A) \right).$$

Note: If $\#N_G(A) \geq \#A \ \forall A$, then max in thm $= 0$, so we get an X-saturating matching. Thus this thm $\Rightarrow$ marriage thm.

Won't give full proof of marriage thm today. Let's do the "easy half"

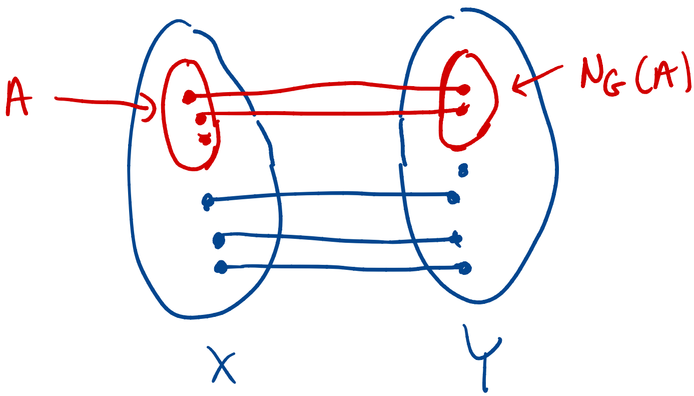
Pf of 1/2 of thm:

Let's prove that max. size of matching is $\leq \#X - \max_{A \subseteq X} (\#A - \#N_G(A))$.

Suffices to prove that for any $A \subseteq X$,

max. matching size $\leq \# X - (\# A - \# N_G(A))$.

So let $A \subseteq X$ be any subset. :



Let's think about how big a matching M can be.

Among $x \in A$, can at most cover $\# N_G(A)$ of them in a matching (by pigeonhole). For $x \in X - A$, could maybe be cover all $\# X - \# A$.

Altogether, size of $M \leq \# N_G(A) + \# X - \# A$. ✓ ▨

On monday we will give pf of other 1/2 of thm, by defining an algorithm to find a maximum matching!

Now let's take a 5 min. break

and when we come back, work on

a matchings worksheet

in breakout groups . . .