Math 4707: Some geometric combinatorics

3/24

Ch. 11 of LPV

Reminders: • HW #4 is due today.

• Midterm #2 is due next week, Wed. 3/31.

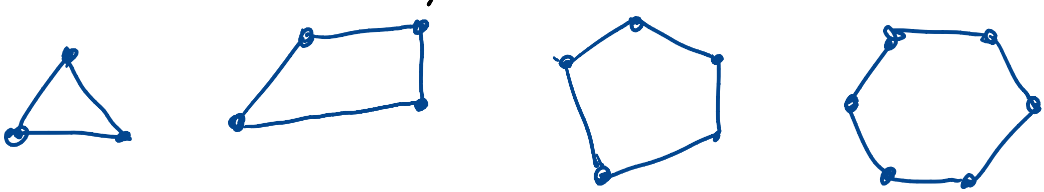
• HW #3 has been graded.

Today we are switching gears. For a while we have been discussing optimization problems and algorithms (e.g. Chapters 9 + 10 in the text). For the remainder of the course, we will discuss connections between geometry and combinatorics (e.g. Ch's 11, 12 + 13).

We start w/ a mishmosh of problems from Ch. 11.

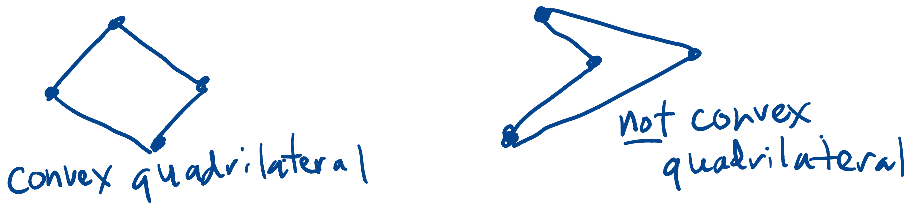
Problem 1: Counting intersections of diagonals in polygons

We all have a pretty good idea of what a polygon is: it's a 2D geometric shape w/ vertices • and straight edges — (or 'sides') like a triangle, quadrilateral, etc.

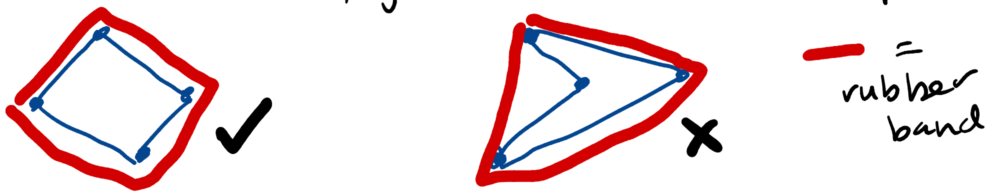


Polygons are sort-of inbetween 'continuous' and 'discrete'.

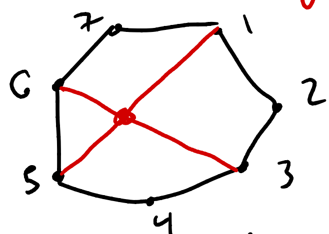
What's a convex polygon? Convex = "doesn't bend in":



Another way to think about convex: if you put a rubber band around it, you form the same shape.

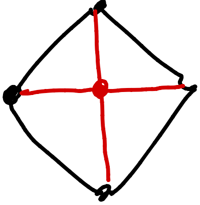


Let's consider a convex n-gon (= n vertices):

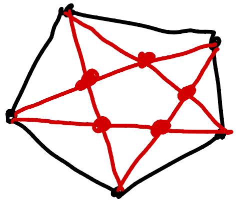


Here have labeled vertices 1 to n clockwise. Also drew 2 diagonals of the polygon (1-5 + 3-6) in red. These diagonals intersect in the marked point.

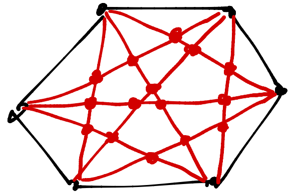
Q: How many intersections between diagonals in a convex n-gon? Let's assume no 3 diag's intersect in a single pt. (can 'wiggle' vert's to make them 'generic'). Let's also only count intersections inside polygon.



$n=4 \to$ 1 intersection

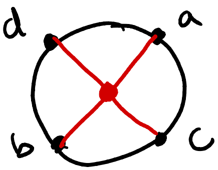


$n=5 \to$ 5 intersections

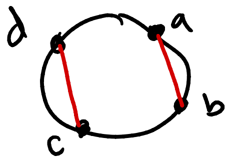


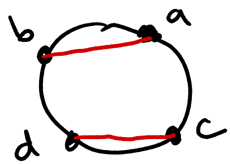
$n=6 \to$ 15 intersections

Idea: Diagonals a-b and c-d intersect
$$\iff \quad a < c < b < d$$



intersect!





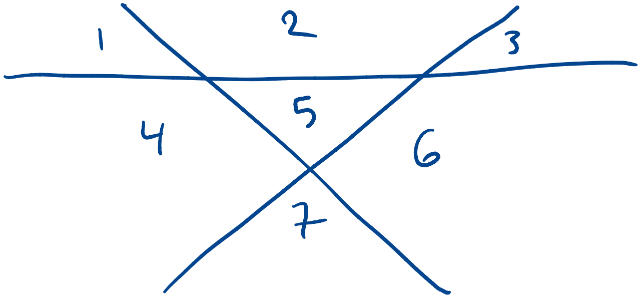
no intersect!

Thus each intersection between diag's corresponds to a unique choice of 4 numbers $a<c<b<d$ from 1 to n.

Thm # of intersections among diagonals of a convex n-gon is $\binom{n}{4} = \dfrac{n(n-1)(n-2)(n-3)}{4!}$.

Problem 2: Counting # of regions n lines divide plane into

If we draw a bunch of lines in the plane, they cut the plane into regions:



In this example, w/ 3 lines we get 7 regions.
How many regions do we get w/ n lines?
Again assume lines are 'generic' (no 3 intersect in 1 pt). + none parallel

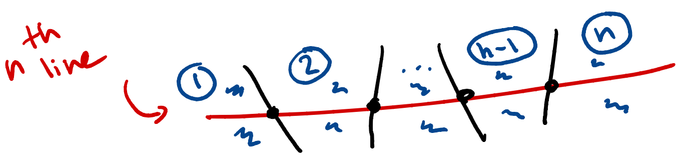
# lines	0	1	2	3	4	...
# regions	1	2	4	7	11	...

If we look at small examples, might think that

$$\#\text{ regions for } n \text{ lines} = \#\text{ regions for } (n-1) \text{ lines} + n \qquad (*)$$

So that $\#\text{ regions} = 1 + (1+2+3+\cdots+n)$.

And this is correct. We can easily prove the recurrence $(*)$ inductively:



n^{th} line will intersect every other line, so it cuts n regions into 2, creating n new regions

<u>Thm</u> # regions that n lines cut plane into

$$= 1 + (1+2+\cdots+n) = 1 + \frac{n(n+1)}{2}.$$

Problem 3: Guaranteeing a convex quadrilateral

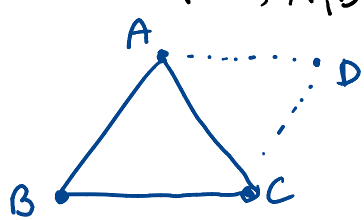
Suppose we randomly select n points in the plane. Can we be sure that among these n points there are 4 that are vertices of a convex quadrilateral?

If $n=4$, we'd be in trouble:

← these don't make convex 4-gon

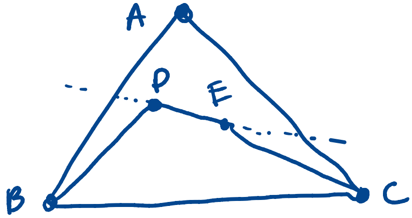
But already for $n=5$, we can be sure: among any 5 points in the plane (in 'general position' = "no 3 on one line"), there are always 4 that make a convex 4-gon.

Pf: The convex hull (= rubber band) of the 5 pts must have at least 3 of the pts, A, B, C, as vertices:



A, B, C form a triangle

If either of the other 2 pts (call them D + E) are outside this triangle, then definitely we can form a convex 4-gon. So assume both inside:



As depicted, let's assume the line thru D+E intersects sides $\overline{AB}$ and $\overline{AC}$ of the triangle (otherwise just relabel pts). Then D, E, C, B form convex 4-gon. $\blacksquare$

Q: How many pts to guarantee convex pentagon?
A: 9. Proof much more complicated.

Thm (Erdős-Szekeres) For any $k \geq 3$, $\exists$ a smallest N_k s.t. $\forall n \geq N_k$, among n pts. in general position, there are k of them forming convex k-gon.

e.g. $N_4 = 5$, $N_5 = 9$, $N_6 = 17$, but exact value unknown for $k > 6$: conjectured to be $1 + 2^{k-2}$.

This thm famous for leading to a marriage. Also one of the 1st results in Ramsey Theory, whose tagline is 'any sufficiently large system contains a big ordered subsystem' or more succinctly,

'complete disorder is impossible.'