Math 4707: More Basic Counting

1/25
Ch.'s 1, 2, 3

Announcements:

- Course website? Canvas? Emails? working?
- HW #1 will be posted by Wednesday, due the following Wed., Feb. 3rd

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Recall: Last class we discussed basic enumeration.

Using the notation $[n] := \{1, 2, 3, \ldots, n\}$ for an n-element set, we explained formulas:

- $\#$ subsets of $[n] = 2^n$
- $\#$ permutations of $[n] = n! = n \cdot (n-1) \cdots 2 \cdot 1$
- $\#$ k-element subsets of $[n] = \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

There were a couple of counting principles that we used to establish these formulas, which might be summarized as...

Multiplication Principle If any object in our collection can be constructed in m steps, where at step i we have exactly k_i choices irrespective of choices made at previous steps then $\#\text{objects} = k_1 \cdot k_2 \cdot \cdots \cdot k_{m-1} \cdot k_m$.

e.g. $\#\text{ subsets of } [n] = 2 \cdot 2 \cdots 2 = 2^n$

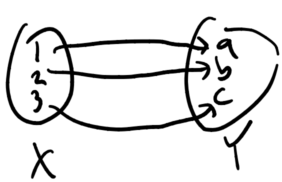
$\#\text{ perm's of } [n] = n \cdot (n-1) \cdots 2 \cdot 1 = n!$

Overcounting principle If every object in set A 'corresponds to'/'makes' ℓ objects in set B, then $\#B = \ell \cdot \#A$ 

e.g., $\#\text{ ordered } k\text{-subsets of } [n] = k! \cdot \#\text{ (unordered) } k\text{-subsets of } [n]$

$$n(n-1)\cdots(n-(k-1)) = k! \cdot \binom{n}{k}$$

Another counting tool we discussed was bijections between sets:



$\Rightarrow \#X = \#Y$

Let's do a few more basic counting problems:

Anagrams How many different rearrangements of the letters in

B A N A N A S

are there? (Don't care if not real words...)

If all the letters were different we'd get $7!$ rearrangements. So let's add colors (or subscripts) to make letters different:

$$B_1 A_1 N_1 A_2 N_2 A_3 S_1$$

For any rearrangement like

A S N A B A N

have $3!$ ways we could color 3 A's

$A_1 \quad A_2 \quad A_3$

$A_1 \quad A_3 \quad A_2 \quad \ldots$

$2!$ ways we can color 2 N's, $1!$ way to color 1 B, and $1!$ way to color 1 S.

overcounting

$\Rightarrow$ # colored rearrangements $= 3! \cdot 2! \cdot 1! \cdot 1! \cdot$ # rearrangements

$\Rightarrow \quad 7! = 3!\,2! \cdot$ # rearrangements

$\Rightarrow \quad$ # rearrangements $= \dfrac{7!}{3!\,2!}$

More generally, ...

Theorem # sequences that are rearrangements of

$$\underbrace{1\,1\,1\cdots 1}_{k_1 \text{ times}}\ \underbrace{2\,2\cdots 2}_{k_2 \text{ times}}\ \cdots\ \underbrace{m\ m\ \cdots\ m}_{k_m \text{ times}}$$

is

$$\frac{n!}{k_1!\,k_2!\cdots k_m!} \quad \text{where } n = k_1 + k_2 + \cdots + k_m.$$

sometimes use notation $\binom{n}{k_1, k_2, \ldots, k_m}$ for this number, called 'multinomial coefficient'

Q: Do we see how the anagrams problem relates to k-element subsets of $[n]$ and the binomial coeff's $\binom{n}{k}$?

Choosing bagel flavors

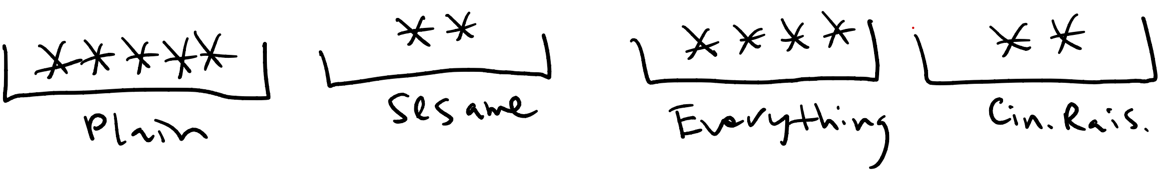
How many ways are there to select 13 bagels, if there are 4 flavors:

plain, Sesame, Everything, Cinnamon Raisin?

e.g., could choose 5 P., 2 S., 4 E., 2 CR.

Useful trick here called 'stars-and-bars'

Represent selection by putting *'s in bins:



Then draw | 's as separators of bins:

$$* * * * * \mid * * \mid * * * * \mid * *$$

How many patterns are there like this?

Those are just anagrams of 13 *'s and 3 |'s.

$$\Rightarrow \# \text{ bagel choices} = \left(\!\binom{13+(4-1)}{13}\!\right).$$

sometimes these are called 'multi-choose numbers'

One more thing related to this basic counting...

Estimation The answers to these counting problems are #'s that grow pretty big as $n \to \infty$, but how big exactly are they?

Q: How many digits in 2^n?

Answer: $\log_{10} 2^n = n \cdot \log_{10} 2 = n \cdot 0.301\ldots$

What about the number $n!$ that pops up in these counting problems?

Stirling's approx:

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

(where $f \sim g$ means $\lim_{n\to\infty} \frac{f}{g} = 1$)

To prove it requires some calculus and so we will not prove it, but you're free to use it on HW/exam problems...

Now after a 5 minute break let's do the worksheet (in breakout groups) which is on poker hand probabilities.

To compute the probability of a hand, just need to know that

$$\text{Prob.}\left(\begin{array}{c}\text{certain}\\ \text{kind of}\\ \text{hand}\end{array}\right) = \frac{\#\text{ that kind of hand}}{\text{total }\#\text{ of hands}}$$

e.g.)

$$\text{Prob}(4\text{ of a kind}) = \frac{\#\text{ hands w/ 4 of a kind}}{\text{total }\#\text{ of hands}}$$

We said last class that total # of 5 card hands from standard 52 card deck $= \binom{52}{5} \approx 2.6$ million

WARNING: Keep ordered vs. unordered information straight here!