Math 4707: Graph coloring

4/12

Ch. 13 of LPV

Reminders: • Midterm #2 has been graded.
You should have decent sense of how you're doing now.

but still 2 more assignments!

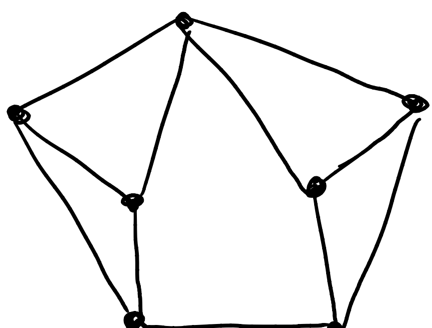
• HW #5 posted, due Wed. 4/21.

• Student Reviews of Teaching now open.

Consider the following 'real world' problem: there are a number of radio towers in a region, and four possible frequencies they could broadcast at; but towers that are close to one another should not be given the same frequency. How to assign frequencies?

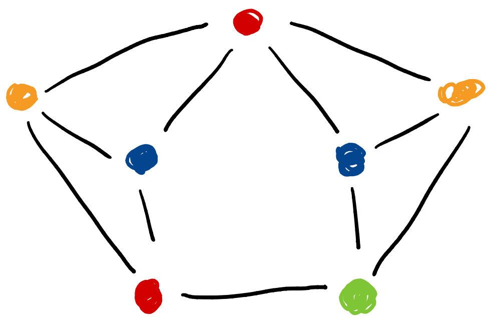
Can represent towers as vertices of a graph:

e.g. 7 towers:



Here edge = towers are close (broadcasts overlap).

Then a valid assignment of frequencies to the towers is the same as a coloring of the vertices of the graph w/ 4 colors such that adjacent vertices are colored differently:

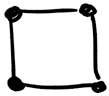
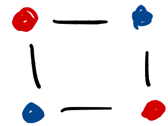


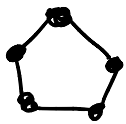
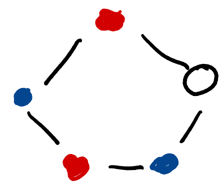
These are the kinds of problems we'll study today (coloring terminology comes from maps, which we'll discuss next class...)

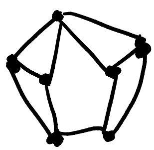
Def'n A (proper) k-coloring of a graph G is an assignment of k colors to its vertices so that adjacent vertices are colored different colors.

The chromatic number $\chi(G)$ is smallest k for which a k-coloring of G exists.

In other words: how *few* colors do we need?

e.g. $\chi($  $) = 2$ since 

$\chi($  $) = 3$ since  ← can't 2-color!

$\chi($  $) \leq 4$ from above coloring

← can show $= 4$

$\chi(K_n) = n$ ← do you see why?

↑ complete graph

Let's try to understand graphs w/ $\chi(G) = K$ for small values of K.

⌈ Note: Since only the adjacency of vertices matters for coloring, let's assume all graphs are simple today. ⌋

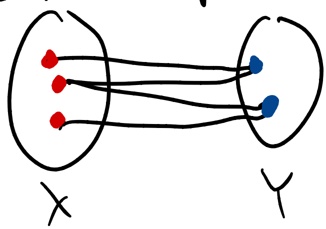
$\chi(G) = 1 \iff G$ has no edges boring!

$\chi(G) = 2$ is very interesting...

In fact, we've seen these graphs before:

Prop: $\chi(G) = 2 \iff G$ is bipartite.
(and G has at least one edge)

Pf: Suppose G has bipartition (X, Y):



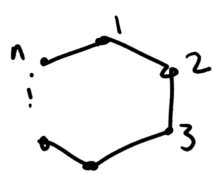
Then can color vertices in X red and in Y blue.

Conversely, a two-coloring gives a bipartition. ▨

We've discussed bipartite graphs a good deal already (remember: matchings)

but: Can we characterize them in some way?

Generalizing above, for the cycle graph

$C_n =$ 

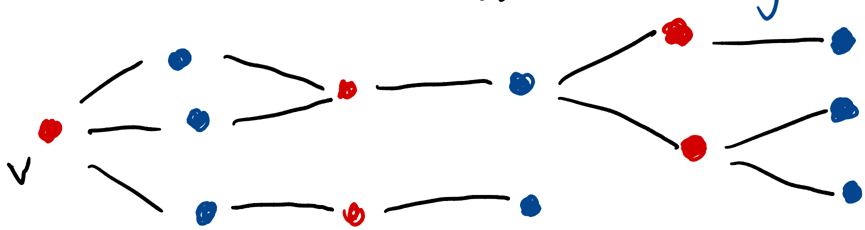
we have
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

It turns out that odd cycles determine whether a graph is bipartite:

Thm G is bipartite $\Leftrightarrow$ G has no odd cycles.

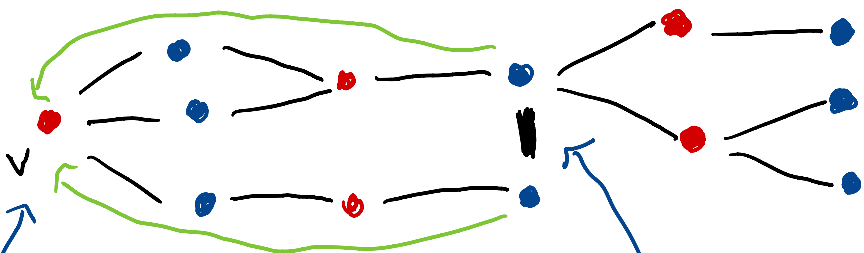
Pf: If G has an odd cycle, then certainly $\chi(G) \geq 3$ since we cannot even two-color that cycle.

So now suppose that G has no odd cycles. Want to show we can 2-color it. So... let's just try: start at any vertex v, and color that vertex red, then color its neighbors blue, then color its neighbors' neighbors red, and so on. We color in "layers" like this:



We can assume G is connected (why?) and thus that we've colored all the vertices of G in this manner. But why is coloring proper?

Suppose two vertices of same color were adjacent:



1st note that these two vertices would have to be at the same layer, otherwise would've colored opposite colors. Then note that if we trace back to the common ancestor in the tree-like structure rooted at v we've built, we find an odd cycle contained in G. Contradiction. ▨

Note: Proof shows how to quickly 2-color any bipartite graph.

Okay, so what about graphs w/ $\chi(G)=3$? Or, 4? Or more? Studying colorings for graphs w/ $\chi(G)\geq 3$ is much harder than for bipartite graphs.

Basic issue: in proof above, we saw that 2-colorings are (essentially) unique. But with k-colorings for $k\geq 3$, in general have many choices.

Of course, there are still some things we can say for k-colorings w/ $k\geq 3$.

Prop. If G contains a subgraph H w/ $\chi(H)=m$, then $\chi(G)\geq m$.

Pf: Takes m colors to even color H. $\blacksquare$

Cor If G has a subgraph isomorphic to complete graph K_m, then $\chi(G)\geq m$.

Of course, there are many graphs w/ $\chi(G) = m$ containing no K_m's (see worksheet).

What about upper bounds on $\chi(G)$?

<u>Prop.</u> Let $\Delta(G)$ denote the maximum degree of G. Then $\chi(G) \le \Delta(G) + 1$.

<u>Pf:</u> By induction. Choose a vertex v, and consider subgraph $G' = G - v$:

By induction, can properly color G' w/ $\Delta(G)+1$ colors. also, v has at most $\Delta(G)$ neighbors, so must be at least one color not used by neighbors of v we can color v. ∎

Again, $\chi(G)$ can be much smaller than $\Delta(G)+1$ (see worksheet).

<u>Rmk:</u> Deciding if G can be 3-colored is a computationally hard problem (like Ham. cycle...) //

Now let's take a 5 min. break...

and when we come back, we

can learn more about coloring

by doing the worksheet

in breakout groups.