Math 4707: More about planar graphs

4/19

Not in LPV

Reminder: • HW#5 is due this Wed. 4/21.

We've officially covered all the material from the textbook that we'll learn about in this course. For the remaining classes we'll discuss some more topics related to Ch.'s 12 + 13: planar graphs and graph coloring. Today we'll talk more abt planar graphs.

Planar Duality

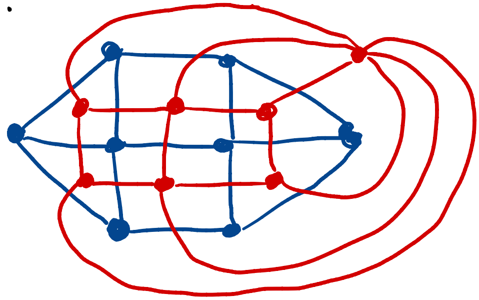
In our discussion of the Four Color Theorem last class, we discussed a way to translate problems about coloring regions of a map to problems about coloring vertices of an associated planar graph.

Let's formalize this procedure + discuss in more detail.

Let G be a connected plane graph (i.e., a planar graph w/ a fixed planar embedding). The dual graph G^* of G has vertices corresponding to faces of G, and when e is an edge on the border of two faces F, F' of

G, we have an edge e^* between corresponding vertices v, v' of G^*:

G



G^*

Observe that

$$\#\text{edges}(G) = \#\text{edges}(G^*)$$
$$\#\text{vertices}(G) = \#\text{faces}(G^*)$$
$$\#\text{faces}(G) = \#\text{vertices}(G^*)$$

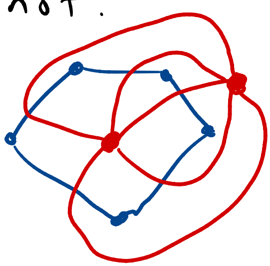
Q: How does this fit in with Euler's formula

$$v + f = e + 2\,?$$

Also notice that $(G^*)^* = G$. (Dual of dual is original graph!)

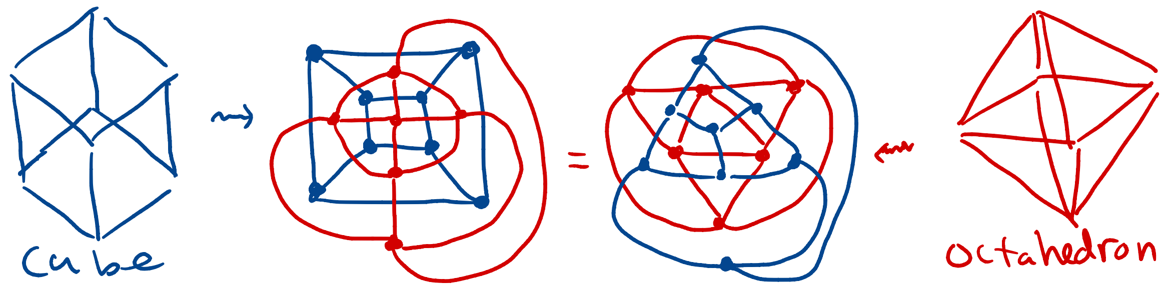
Rmk: Previously when discussing planar graphs, we'd been assuming they were simple. But G^* might have multiple edges or loops, even if G does not:

$G \cong C_5$

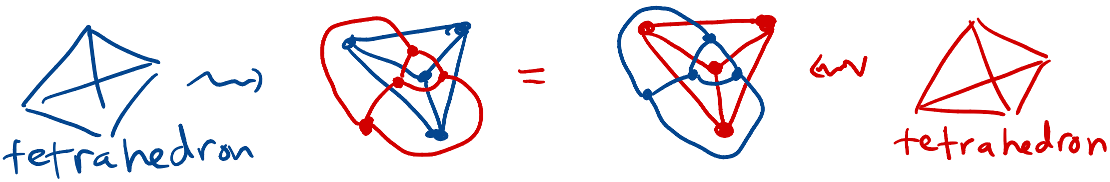


$G^* =$ graph w/ 5 edges between 2 vertices

Planar duality was first discovered in the context of **convex polyhedra**. Remember how we associated to each convex polyhedron a planar graph G? The polyhedron associated to G^* is called its **(polar) dual polyhedron**. For example, the cube and octahedron are dual:

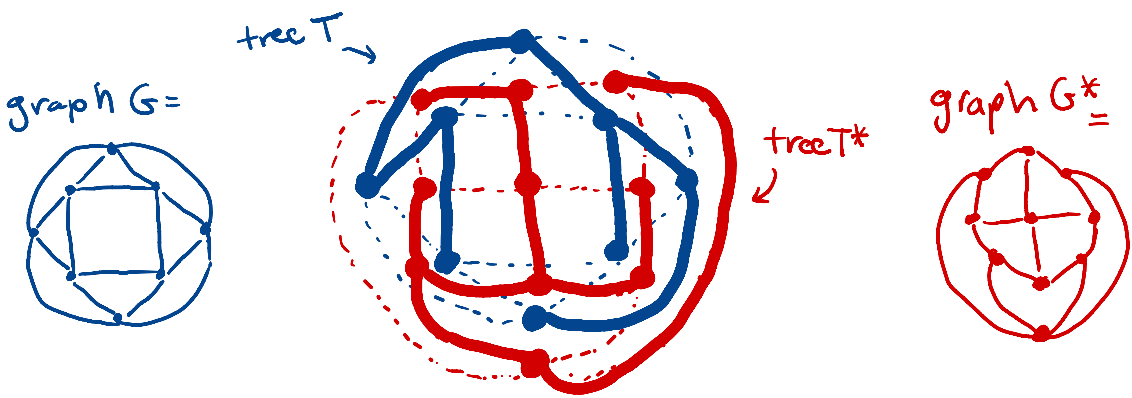


Similarly, the dodecahedron and icosahedron are dual (see today's worksheet...). What about the tetrahedron? We call a planar graph G **self-dual** if it's isomorphic to its dual G^*:



Tetrahedron ($=K_4$) is self-dual.

Another remarkable fact: for any connected planar graph G, G and G^* have the same # of spanning trees.
Can define a bijection $T \mapsto T^*$ from spanning trees of G to G^* as follows: let T^* consist of all edges of G^* that don't cross any edge of T:



Not hard to check that T^* is a spanning tree of G^* and that $T \mapsto T^*$ is a bijection.
[Left as exercise for you.]

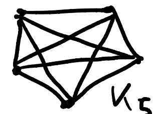
Note that

every edge in G is either in T or crossed by T^*

$$\#\text{edges}(G) = \#\text{edges}(T) + \#\text{edges}(T^*)$$
$$= (\#\text{vertices}(G) - 1) + (\#\text{vertices}(G^*) - 1)$$
$$= \#\text{vertices}(G) - 1 + \#\text{faces}(G) - 1,$$

which gives another proof of Euler's formula.

Kuratowski's Theorem



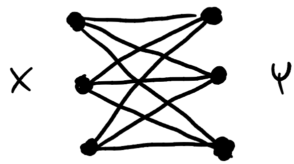
K_5

We've seen one basic example of a non-planar graph: K_5. Recall how we proved K_5 is not planar: it violates the inequality

$$\#\text{edges}(G) \le 3 \cdot \#\text{vertices}(G) - 6$$

that holds for all (simple) planar graphs G.
(And recall that this was a corollary of Euler's formula.)

There's another basic non-planar graph: $K_{3,3}$, the complete bipartite graph w/ 2 parts of size 3:



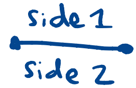
$K_{3,3}$

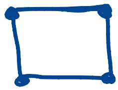
Of course, to prove $K_{3,3}$ is not planar, not enough to draw it with crossings. And $\#\text{edges}(K_{3,3}) = 9 \le 3 \cdot \#\text{vert's}(K_{3,3}) - 6$, So the same argument for K_5 doesn't work. But...

Lemma If G is a connected, (simple) bipartite planar graph,

$$\#\text{edges}(G) \le 2 \cdot n - 4,$$

where $n = \#\text{vertices}(G)$ (for $n \ge 3$).

Pf: Think of each edge having two sides: 

Since G is bipartite, smallest # of sides in each face is $\underline{4}$:  (What abt outer face?? This is why $n \geq 3$ needed.)

was 3 for arbitrary G

So
$$\#\text{faces} \leq \frac{1}{4} \cdot \#\text{sides} = \frac{2}{4} \cdot \#\text{edges} = \frac{1}{2}\#\text{edges}.$$
Combined w/ Euler's formula $e + 2 = f + v$, gives
$$\#\text{edges} + 2 = \#\text{faces} + \#\text{vert's} \leq \frac{1}{2}\#\text{edges} + \#\text{vert's},$$
and via algebra we get:
$$\#\text{edges} \leq 2 \cdot n - 4.$$
$\boxtimes$

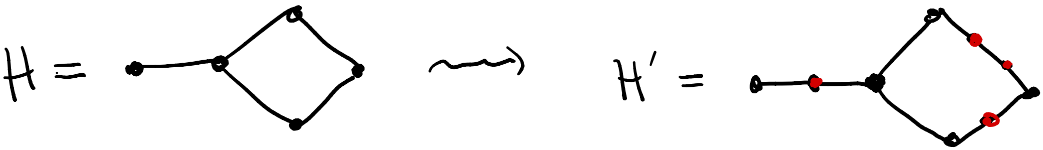
Cor $K_{3,3}$ is not planar.

Pf: Have
$$\#\text{edges}(K_{3,3}) = 9 > 8 = 2 \cdot 6 - 4$$
in contradiction w/ previous lemma. $\boxtimes$

It turns out that K_5 and $K_{3,3}$ are the "only obstructions" to a graph being planar, in a sense made precise by Kuratowski's theorem.

To state this thm, we need the notion of a subdivision of a graph. For a graph H and an edge e of H, subdividing e means "putting a new vertex in the middle of e":

A **subdivision** H' of H is any graph H' obtained from H by repeatedly subdividing edges:



It should be pretty clear that

Prop: If H' is a subdivision of H, then

$$H' \text{ is planar} \iff H \text{ is planar.}$$

So for example, any subdivisions of K_5 or $K_{3,3}$ are <u>not</u> planar. And K's thm says...

<u>Thm</u> (Kuratowski's Theorem)

A graph G is planar if and only if it has no subgraph H isomorphic to a subdivision of K_5 or $K_{3,3}$.

One direction (G planar $\Rightarrow$ no subgraph H can be subdivision of K_5 / $K_{3,3}$)

we've already explained. Other direction is <u>harder</u> and we will not prove it...

Now let's take a 5 min. break...

and when we come back

we can do a few problems

about planar graphs

on today's worksheet

in breakout groups.