Math 4707: The Tutte polynomial

4/28
Not in LPV

Reminder: • Final due in 1 week on Wed., 5/5.

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In the past two classes we've seen deletion-contraction is a very powerful tool for understanding graphs. Today we'll see "how far" you can go w/ deletion-contraction. It turns out there is a universal deletion-contraction invariant, called the Tutte polynomial.

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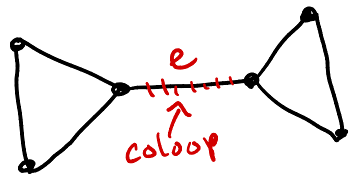
Before we define Tutte polynomial, need to slightly modify our definition of contraction. Previously we were always working w/ simple graphs, but now it's important to allow multiple edges and loops. So now when we contract an edge e, we do not remove multiple edges:

And if we contract a multiple edge, we create loops:

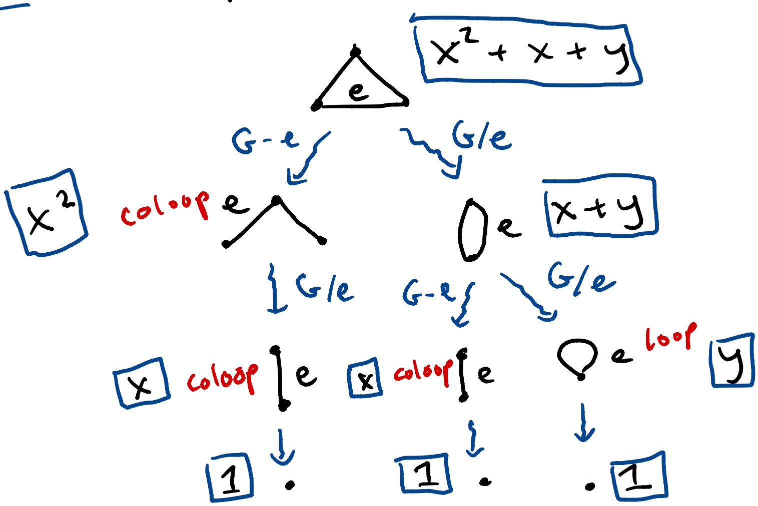
We need one more notion to define Tutte poly.:
an edge e of G is a **coloop** a.k.a. **isthmus** if deleting e increases the # of connected components:

e.g. $G =$ 

Def'n The **Tutte polynomial** of G, denoted $T_G(x,y)$, is the unique polynomial in two variables x and y s.t.:

- $T_G(x,y) = T_{G-e}(x,y) + T_{G/e}(x,y)$ for any edge e which is not a loop or coloop.
- $T_G(x,y) = x\, T_{G/e}(x,y)$ if e is a coloop.
- $T_G(x,y) = y\, T_{G-e}(x,y)$ if e is a loop.
- $T_G(x,y) = 1$ if G has no edges.

e.g. to compute $T_G(x,y)$ for $G = K_3$ triangle:



Here we write $\boxed{T_G(x,y)}$ next to graph G

Rmk: Implicit in def'n of $T_G(x,y)$ is the fact that it does not matter which order we delete/contract the edges in. Actually, this is a Theorem, which can be proved using a different def'n of $T_G(x,y)$.

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Essentially by definition, the Tutte polynomial is the universal deletion-contraction invariants: we get others via specialization. For instance...

Prop. The chromatic polynomial of G is

$$\chi(G,k) = (-1)^{\#V - c(G)} k^{c(G)} T_G(1-k, 0)$$

where $\#V = \#$vert's of G and $c(G) = \#$ components of G.

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e.g. For $G = K_3$ we saw before that

$$\chi(G,k) = k(k-1)(k-2)$$

and $(-1)^{3-1} k^1 T_G(1-k,0) = k((1-k)^2 + (1-k) + 0) = k(k-1)(k-2)$ ✓

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The proof of prop. is same as we have seen before: show they satisfy same recurrence.

Similarly, from last class we get...

Prop. $T_G(2,0) = \#$ acyclic orientations of G.

Another very important specialization of Tutte poly. is $x = y = 1$.

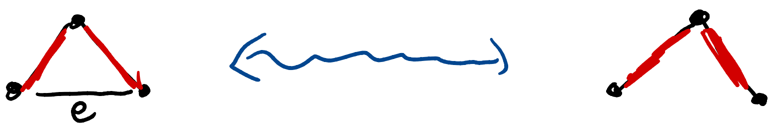
Prop. For a connected graph G,
$T_G(1,1) = \#$ spanning trees of G.

Pf: Suffices to show recurrence

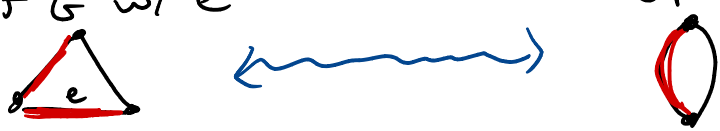
$$\#\text{span.tree}(G) = \#\text{span.tree}(G-e) + \#\text{span.tree}(G/e)$$

for a non-loop/coloop e. We explained why this is many classes ago... there are bijections:

spanning trees of G w/out e $\Longleftrightarrow$ spanning trees of $G-e$



spanning trees of G w/ e $\Longleftrightarrow$ spanning trees of G/e



$\boxtimes$

e.g. $\#\text{span.tree}(K_3) = 3 = (x^2+x+y)\big|_{x=y=1}$. ✓

Let's finish w/ a cool symmetry of $T_G(x,y)$:

Thm Suppose G is planar graph, G^* its dual.
Then $T_G(x,y) = T_{G^*}(y,x)$.

In other words, planar duality swaps x and y!

Basic idea of pf:

deletion of e in G $\Leftrightarrow$ contraction of e^* in G^*

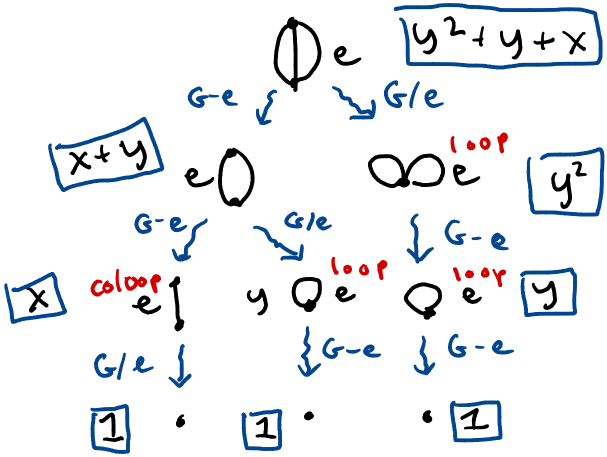
e is loop in G $\Leftrightarrow$ e^* is coloop in G^*.

Rmk: We've already seen #span.tree(G) = #sp.tree(G^*).
Since $T_G(1,1) = T_{G^*}(1,1)$, the tutte poly. gives another proof of this.

e.g. for $G =$ , $G^* =$  and $T_{G^*}(x,y)$ is ...

$\boxed{y^2+y+x} = T_G(y,x)$

Here we write $\boxed{T_G(x,y)}$ next to graph G



Now let's take a 5 min. break . . .

and when we come back we can do the final worksheet of the semester, on the Tutte poly.,

in breakout groups.