# Math 4707: Pigeonhole Principle + Principle of Inclusion-Exclusion

1/27

Ch. 2 LPV

**Announcements:**

- HW #1 has been posted, is due in a week, Wed. Feb. 3rd

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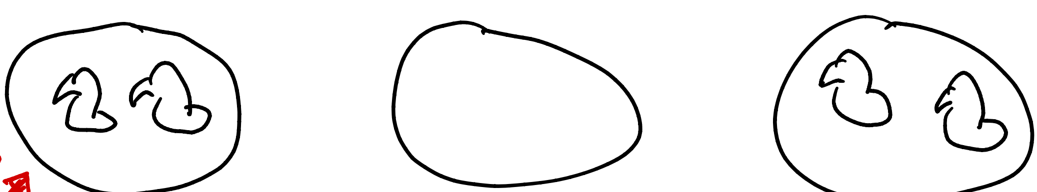
## Pigeonhole Principle

The **pigeonhole principle** is a simple but very powerful tool for understanding discrete structures, like **induction**, which we already discussed, and the **principle of inclusion-exclusion**, which we'll discuss soon.

(It's not very related to counting, so it's a bit of an aside from what we've been doing...)

**Thm (Pigeonhole Principle)** If you put $m+1$ pigeons into $m$ holes, at least one hole has at least two pigeons.

$4 > 3$



Pf: Suppose each hole has at most one pigeon.
Then # pigeons = # holes that have a pigeon
$\leq$ # holes.
But by assumption, # pigeons > # holes, contradiction. $\boxtimes$

There is also a variant if we want to force more than two pigeons in same hole:

Thm Suppose that you have $r \cdot m + 1$ pigeons and $m$ holes, then there is a hole that has at least $r+1$ pigeons in it.

Pf: Same! $\boxtimes$

Pigeonhole principle seems completely trivial, but it's very useful! Trick is: choosing the holes! Let's show some example applications...

e.g. Show that if you 21 numbers between 1 and 100, there are two of them whose difference is $\leq 4$.

**Pf:** Divide 1 - 100 into groups of 5 consec. #'s:

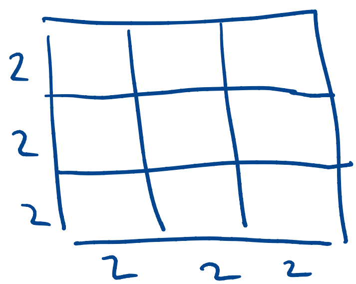
1 2 3 4 5 | 6 7 8 9 10 | . . . | 96 97 98 99 100

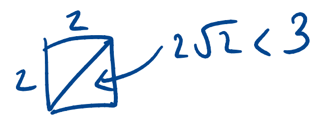
Those are our holes and the #'s are our pigeons.

Since 20 holes + 21 pigeons $\Rightarrow$ 2 numbers in same gp. of 5, they have difference at most 4. ✓ $\blacksquare$

**e.g.** You pick 10 points in a 6in × 6in square. Show that there are 2 pts within 3 in of each other.

**Pf:** Divide up 6×6 square into 3×3 grid of 2×2 sq's:




9 2×2 sq's and 10 pts $\Rightarrow$ One 2×2 sq has two pts in it. Max. distance in 2×2 sq. is diagonal of len. $2\sqrt{2} < 2(1.5) = 3$. ✓ $\blacksquare$

Remark: P.P. tells you there are some two pigeons in some hole together, but doesn't help you find them! "Pure existance result"

**Birthday "paradox"** Suppose we have a group of $k$ people. Say two of them are twins if they share the same birthday (out of $N=365$ days)

**Q:** How big does $k$ need to be to guarantee twins?

**ANSWER:** 366 (or 367 if we're worried abt. leap day...)

Let's see if we have any twins right now!

**"Paradox":** At only $k=24$ people, $\geq 50\%$ chance of twins.

To see this, note:

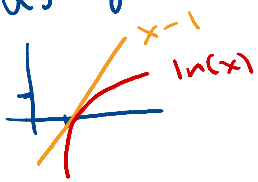
$$\Pr(\text{No twins}) = \frac{N(N-1)(N-2)\cdots(N-(k-1))}{N^k}$$ why?

To understand this... take logs!

$$\ln(\Pr(\text{no twins})) = \ln\left(\frac{N(N-1)\cdots(N-(k-1))}{N^k}\right)$$

$$= \sum_{j=0}^{k-1} \ln\left(\frac{N-j}{N}\right)$$ $\ln(a\cdot b) = \ln(a) + \ln(b)$

using $\ln(x) \leq x-1$




$$\leq \sum_{j=0}^{k-1} \frac{N-j}{N} - 1 = \frac{-\sum_{j=0}^{k-1} j}{N} = \frac{-k(k-1)}{2N}$$

w/ $k=24, N=365 \Rightarrow \Pr(\text{no twins}) \leq e^{\frac{-24(23)}{2\cdot 365}} \sim e^{-0.75} \sim 46\%$

w/ $k=70, N=365 \Rightarrow \Pr(\text{no twins}) \leq 1\%$ ← WOW!

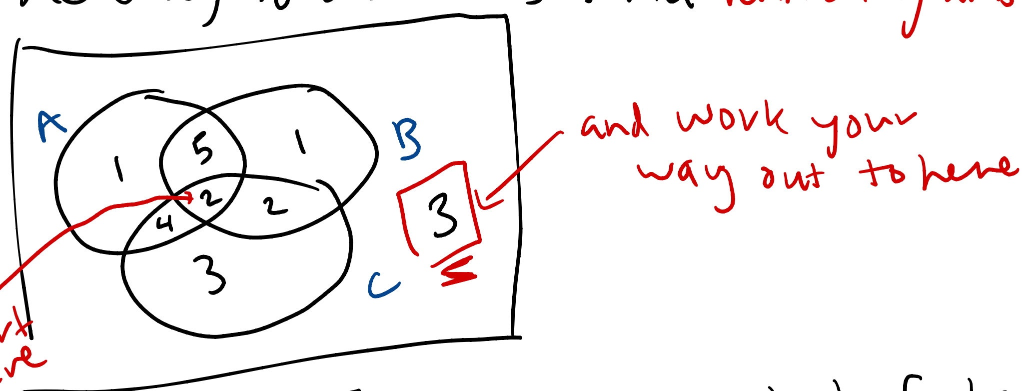
# Principle of Inclusion-Exclusion (P.I.E.)

Now let's get back to counting with a very important tool called the Principle of Inclusion-Exclusion

Suppose that in a class, some students like Art, some like Bicycling, and some like Cooking. There are 21 students in the class, 12 like Art, 10 like Biking, 11 like cooking, 7 like A + B, 6 like A + C, 4 like B + C, and 2 like all 3. How many like none of A, B, or C?

Ans: 3.

One way to do this is via Venn diagrams:




But the P.I.E. gives a somewhat faster/more compact answer...

Letting $S$ be set of all students...

students not in $A, B,$ or $C$

$$\#(S - A \cup B \cup C) = \#S - \#A - \#B - \#C$$

start w/ everything

exclude sets we don't want

$$+ \#A \cap B + \#A \cap C + \#B \cap C$$

we 'double-subtracted' $A \cap B$, so re-include it!

$$- \#A \cap B \cap C$$

we included $A \cap B \cap C$ too much! re-exclude it

$$= 21 - 12 - 10 - 11 + 7 + 6 + 4 - 2$$

$$= (3 + 1 + 1 + 3 + 5 + 4 + 2 + 2)$$
$$- (1 + 5 + 4 + 2) - (1 + 5 + 2 + 2) - (3 + 4 + 2 + 2)$$
$$+ (5 + 2) + (4 + 2) + (2 + 2) - 2 = 3$$

More abstractly... the P.I.E. is:

**Thm** Let $U$ be a set and $A_1, A_2, \dots, A_k \subseteq U$

Then, $\#(U - A_1 \cup A_2 \cup \cdots \cup A_k)$

$$= \#U - \#A_1 - \#A_2 - \cdots - \#A_k + \#A_1 \cap A_2 + \#A_1 \cap A_3$$
$$+ \cdots - \#A_1 \cap A_2 \cap A_3 - \cdots (\pm) \#A_1 \cap A_2 \cap \cdots \cap A_k$$

We'll (probably) prove the P.I.E. next class.
Hopefully it should be somewhat intuitive.
Let's do some more examples...

## Derangements

A permutation $w = w_1 w_2 \dots w_n$ of $[n]$ is called a **derangement** if $w_i \neq i \ \forall i$.

e.g. 2143 is a derangement

3 $\cancel{2}$ 4 1 is **not** a derangement.

**Q:** How many derangements of $[n]$ are there?

e.g. $n=2$ 21 | $n=3$ 312, 231

$n=4$ ... ???

Let's try to use P.I.E. to count derangements by **excluding bad permutations**...

# derangements ($w_i \neq i \ \forall i$)

$= \text{total } \# w\text{'s} - \# w\text{'s w/ } w_1 = 1 - \# w\text{'s w/ } w_2 = 2$

$- \cdots - \# w\text{'s w/ } w_n = n + \# w\text{'s w/ } w_1 = 1$ **AND** $w_2 = 2$

$+ \cdots - \cdots \pm \cdots$

$$= n! - \sum_{k=1}^{n} \sum_{\substack{S \subseteq [n], \\ \# S = k}} (-1)^k \cdot \underbrace{\# \text{ perm's w/ } w_i = i \ \forall i \in S}_{= (n-k)!, \text{ since can permute } j \notin S \text{ arbitrarily!}}$$

$$= n! - \sum_{k=1}^{n} (-1)^k \cdot \binom{n}{k} \cdot (n-k)!$$

$\binom{n}{k}$ ← $\#\{S \subseteq [n] : \# S = k\}$

$$= n! - \sum_{k=1}^{n} (-1)^k \frac{n!}{k! \cancel{(n-k)!}} \cdot \cancel{(n-k)!} = n! - \sum_{k=1}^{n} (-1)^k \frac{n!}{k!}$$

$$= \boxed{\sum_{k=0}^{n} (-1)^k \frac{n!}{k!}}$$

(note $0! = 1$)

← nice answer!

e.g. $n = 2 \Rightarrow 2!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!}\right) = 2\left(\frac{1}{2}\right) = 1$ ✓

$n = 3 \Rightarrow 3!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 3!\left(\frac{1}{2} - \frac{1}{6}\right) = 3 - 1 = 2$ ✓

$n = 4 \Rightarrow 4!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 12 - 4 + 1 = 9$ ✓

# Hat-check problem

100 people attend a play and they all check their hats at the lobby. The attendant forgets to record names for hats, and so at the end of the night returns hats at random. What's the probability no one gets their own hat back?

ANS: This is just asking: what's the prob. a random permutation of $[n]$ is a derangement?

So
$$= \frac{\#\text{ derangements}}{\#\text{ perm's}} = \frac{\sum_{k=0}^{n} (-1)^k \frac{n!}{k!}}{n!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} = *$$

how to estimate this?

Recall: $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

So $* \approx e^{-1} = \frac{1}{e} = 36.787\ldots\%$

(Note: doesn't really depend on what $n=100$ is!)

Now let's take a 5 minute break...

and in any remaining time we can work in breakout groups on the worksheet for today which has several more problems about the P.I.E.