Math 4707: Fibonacci #'s and generating functions

2/8

Ch. 4 of LPV

Reminder: • HW#2 will be posted by Wednesday, due next Wed., 2/17

• Working on grading HW #1.

Fibonacci numbers

We've already seen several famous sequences of combinatorial numbers (e.g. the binomial coefficients and the Stirling #'s). Today and next class we will study some more famous numbers.

Leonardo Fibonacci, 13th century Italian mathematician, posed the following problem:

- Rabbits reproduce in their 2nd month of life, and every month thereafter. If a farmer starts with a newborn rabbit in the 1st month, how many rabbits will he have in the 10th month?

Let F_n = # rabbits on n^{th} month

$F_1 = 1$ | $F_6 = 5+3 = 8$
$F_2 = 1$ | $F_7 = 8+5 = 13$
$F_3 = 1+1 = 2$ | $F_8 = 13+8 = 21$
$F_4 = 2+1 = 3$ | $F_9 = 21+13 = 34$
$F_5 = 3+2 = 5$ | $F_{10} = 34+21 = 55$

$$F_n = \underbrace{F_{n-1}}_{\text{rabbits alive last month}} + \underbrace{F_{n-2}}_{\text{newborn rabbit for each rabbit at least 2 mo. old}} \quad (*) \qquad n \geq 3$$

The **Fibonacci numbers** F_n are uniquely determined by this recurrence relation (*) together with the initial conditions $F_1 = 1$ and $F_2 = 1$.

Q: How many ways are there to write n as a sum of 1's and 2's?
(order matters!)

e.g. $n=1$ $\quad 1=1 \quad$ 1 way

$n=2$ $\quad 2=2,\ 2=1+1 \quad$ 2 ways

$n=3$ $\quad 3=1+1+1,\ 3=1+2,\ 3=2+1 \quad$ 3 ways

$n=4$ $\quad 1+1+1+1,\ 1+1+2,\ 1+2+1,\ 2+1+1,\ 2+2 \quad$ 5 ways

Conj. $\underset{\text{to write } n}{\#\text{ways}} = F_{n+1}$

Pf: Have recurrence

$\underset{\text{as sum of 1's and 2's}}{\#\text{ways to write } n} = \underset{\text{to write}}{\#\text{ways}} \; n-1 + \underset{\text{write}}{\#\text{ways to}} \; n-2$

which can be proven bijectively:

$$n = a_1 + a_2 + \cdots + a_k \mapsto \begin{cases} n-1 = a_1 + \cdots + a_{k-1} & \text{if } a_k = 1, \\ n-2 = a_1 + \cdots + a_{k-1} & \text{if } a_k = 2. \end{cases}$$

Then need to check initial conditions. ▨

$F_2 = 1 = \#\text{ways } n=1$ $F_3 = 2 = \#\text{ways } n=2$ ✓

Aside: In Sanskrit poetry, there are 2 kinds of syllables: short (=1 measure), long (=2 measures)

Q: How many syllabic patterns are there when we have n measures?

A: F_{n-1} (same as 1's and 2's problem)

Ancient Indian mathematicians (e.g. Pingala c.300 BCE) studied this problem. Fibonacci #'s are an example of "Stigler's law of eponomy".

Patterns in Fibonacci #'s

Just as w/ Pascal's Δ of $\binom{n}{k}$, there are many patterns involving Fibonacci #'s. For example

Prop. $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

(Here $F_0 = 0$, a useful convention.)

Pf: By induction, using the recurrence.

Base cases: $n=0$ $\quad 0 = 1-1$ ✓

$n=1$ $\quad 0+1 = 2-1$ ✓

Induction Step: $(F_0 + F_1 + \cdots + F_{n-1}) + F_n =$

$(F_{n+1} - 1) + F_n =$ (by induction)

$F_{n+1} + F_n - 1 = F_{n+2} - 1$ (by recurrence for F_{n+2}) $\blacksquare$

Many other patterns, e.g.

$$F_n^2 + F_{n-1}^2 = F_{2n-1}$$

can be proved similarly, using induction.

See the textbook and/or HW #2 . . .

Another interesting fact about Fib. #'s is

Theorem (Zeckendorf)

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci #'s (w/out F_0 + F_1)

e.g., $64 = \underset{F_{10}}{\underset{\parallel}{55}} + \underset{F_6}{\underset{\parallel}{8}} + \underset{F_2}{\underset{\parallel}{1}}$

but not $64 = \underset{F_9}{\underset{\parallel}{34}} + \underset{F_8}{\underset{\parallel}{21}} + \underset{F_6}{\underset{\parallel}{8}} + \underset{F_2}{\underset{\parallel}{1}}$

Compare this to...

Theorem (Binary representation)

Every positive number can be written uniquely as a sum of powers of 2.

e.g., $11 = 8 + 2 + 1 = 2^3 + 2^1 + 2^0$

—

Leads us to wonder...

Q: How fast do the Fib. #'s grow?

Generating functions!

To answer the Q of how fast F_n, we will use a very powerful tool called generating functions. (Note: the book doesn't do this...)

Def'n If a_n, $n \geq 0$ is a sequence of #'s, its generating function is

$$A(x) := \sum_{n \geq 0} a_n x^n.$$

You can either think of this as a formal expression (a power series) or a function of the parameter x (e.g., $x \in \mathbb{R}$ or $x \in \mathbb{C}$).

E.g. If $a_n = 2^n$ $\forall n$, then setting

$$A(x) := \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

we have $A(x) = \frac{1}{1-2x}$, because in general for a geometric series we know:

$$\boxed{1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.} \quad r<1$$

Let's now form the gen. fn. for Fib. #'s:

$$F(x) = \sum_{n\geq 0} F_n x^n = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \dots$$

The recurrence (*) let's us write:

$$F(x) = \sum_{n\geq 0} F_n x^n = 0 + x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n$$

$$= x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n$$

$$= x + \sum_{n\geq 1} F_n x^{n+1} + \sum_{n\geq 0} F_n x^{n+2}$$

$$= x + xF(x) + x^2 F(x)$$

So

$$-x^2 F(x) - xF(x) + F(x) = x$$

$$\Rightarrow \boxed{F(x) = \frac{x}{1-x-x^2}.}$$

//

OK... but so what? We found a closed expression for $F(x)$, but what does that tell us about the #'s F_n?

Actually, ... it tells us a lot!

Recall that the geometric series formula says that $\sum_{n\geq 0} c^n x^n = \frac{1}{1-cx}$.

But how is that useful for the Fib #'s w/ $F(x) = \frac{x}{1-x-x^2}$?

Well first, let's observe

$$1 - x - x^2 = \Big(1 - \underbrace{\tfrac{1+\sqrt{5}}{2}}_{\phi}\, x\Big)\Big(1 - \underbrace{\tfrac{1-\sqrt{5}}{2}}_{\psi}\, x\Big)$$

How did I find this... ?

So $F(x) = \frac{x}{(1-\phi x)(1-\psi x)}$, but still don't see the connection to geometric series until we remember Partial fractions:

$$\frac{x}{(1-\phi x)(1-\psi x)} = \frac{A}{1-\phi x} + \frac{B}{1-\psi x}$$

$$\Rightarrow \quad x = (1-\psi x)A + (1-\phi x)B$$
$$= (A+B)1 + (-\psi A - \phi B)x$$

$$\Rightarrow \quad A+B=0, \quad -\psi A - \phi B = 1$$

$$\ldots \Rightarrow \quad A = \frac{1}{\sqrt{5}}, \quad B = \frac{-1}{\sqrt{5}}.$$

So finally,

$$\sum_{n\ge 0} F_n x^n = F(x) = \frac{1/\sqrt{5}}{1-\phi x} - \frac{1/\sqrt{5}}{1-\psi x}$$

$$= \frac{1}{\sqrt{5}} \sum_{n\ge 0} \phi^n x^n - \frac{1}{\sqrt{5}} \sum_{n\ge 0} \psi^n x^n$$

So extracting coefficient of x^n,

exact formula!

$$F_n = \frac{1}{\sqrt{5}}\Big(\underbrace{\frac{1+\sqrt{5}}{2}}_{1.618\ldots}\Big)^n - \frac{1}{\sqrt{5}}\Big(\underbrace{\frac{1-\sqrt{5}}{2}}_{-0.618\ldots}\Big)^n$$

In particular, $F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ as $n\to\infty$

Hopefully starting to see power of gen. fn's!

Now let's take a 5 min. break, and when we're done we will practice using generating functions on today's worksheet in breakout groups!