Math 4707: Catalan numbers + more generating fn's

2/10

not in textbook!

Reminder: • HW #2 has been posted, due in one week, on 2/17

Today we'll continue talking about famous combinatorial sequences of #'s by introducing the Catalan numbers. Very popular topic: e.g., R. Stanley has a book called "Catalan numbers" with ≥ 200 interpretations!!

First let's go over something from last class's worksheet

Recall from calculus...

Thm (Taylor Series)

For a 'reasonable' function $f: \mathbb{R} \to \mathbb{R}$, have

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!},$$

where $f^{(k)} = k^{\text{th}}$ derivative of f.

Let's take $f(x) = (1+x)^n$, where $n \in \mathbb{R}$ is any real number

e.g. $(1+x)^{-3} = \frac{1}{(1+x)^3}$, $(1+x)^{\frac{1}{2}} = \sqrt{1+x}$, $(1+x)^{\pi} = ???$

Remember from calculus that $f'(x) = n(1+x)^{n-1}$, and

$f^{(k)}(x) = n \cdot (n-1) \cdots (n-(k-1)) (1+x)^{n-k}$, so

Thm (Generalized binomial theorem)

For any $n \in \mathbb{R}$, $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$, where

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-(k-1))}{k!}.$$ ← generalized def. of binomial coeff.'s.

NOTE: If $n \in \mathbb{N}$ is a nonnegative integer, then

$\binom{n}{k} = 0$ when $k > n$, so we get as usual

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k. \checkmark$$

On the worksheet, it asked you to consider taking n to be a negative integer, e.g. $(1+x)^{-4} = \frac{1}{(1+x)^4}$.

$$\frac{1}{(1-x)^4} = \sum_{k=0}^{\infty} \binom{-4}{k} (-x)^k = \sum_{k=0}^{\infty} \frac{-4(-4-1)\cdots(-4-(k-1))}{k!} (-x)^k$$

$$\begin{matrix} \| \\ (1+x+x^2+\cdots) \\ (1+x+x^2+\cdots) \\ (1+x+x^2+\cdots) \\ (1+x+x^2+\cdots) \end{matrix} \quad = \sum_{k=0}^{\infty} (-1)^k \frac{(4+(k-1))\cdots(4+1)(4)}{k!} (-x)^k = \sum_{k=0}^{\infty} \binom{4+k-1}{k} x^k$$

do we see connection to 4 flavors of bagels problem?

Gives another proof for 'multichoose' formula.

Now think about when n is a rational number:

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\cdots\left(-\frac{(2k-1)}{2}\right)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1\cdot 3\cdot 5\cdot \cdots (2k-1)}{2^k \; k!} x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1\cdot 3\cdots(2k-1)}{2^k k!} \frac{2\cdot 4\cdot 6\cdots(2k)}{2^k \; k!} x^k$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} \left(-\frac{1}{4}\right)^k x^k$$

So... $(1-4x)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(-\frac{1}{\cancel{4}}\right)^k (-\cancel{4}x)^k = \sum_{k=0}^{\infty} \binom{2k}{k} x^k,$

the g.f. of central binomial coeff's!

$\binom{2k}{k} = 1, 2, 6, 20, 70, \ldots$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
⋮

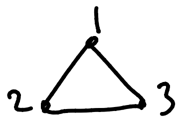
Rmk The g.f.'s we discussed earlier were all rational, i.e., ratios $\frac{P(x)}{Q(x)}$ of polynomials P, Q.

$(1-4x)^{-1/2} = \frac{1}{\sqrt{1-4x}}$ is not rational (it's algebraic).

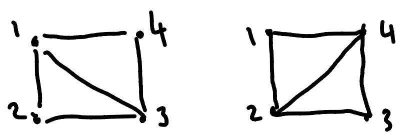
Now let's consider a new counting problem...

$C_n :=$ # triangulations of a $(n+2)$-gon.

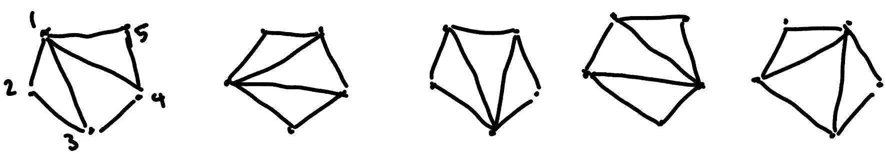
$C_1 = 1$



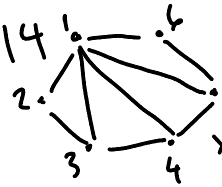
$C_2 = 2$



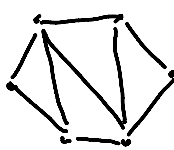
$C_3 = 5$



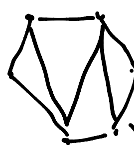
$C_4 = 14$



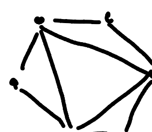
$\times 6$ rotations



$\times 3$



$\times 3$



$\times 2$

$C_5 = 42$ ... no way I'm drawing those!

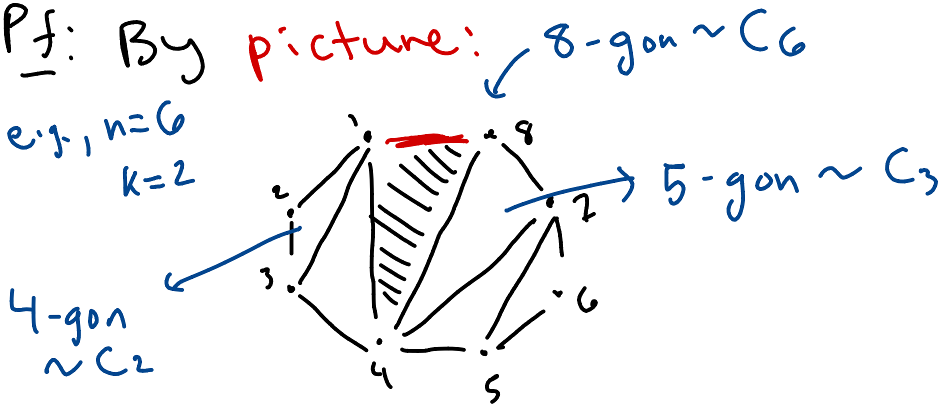
Also reasonable to define $C_0 = 1$

The C_n are called Catalan numbers.

Thm (Fundamental recurrence)

For $n \geq 1$, $C_n = \sum_{k=0}^{n-1} C_k \, C_{(n-1)-k}$.

Pf: By picture:



"base" edge triangle splits any triangulation of an $(n+2)$-gon ($\sim C_n$) into tri. of $(k+2)$-gon ($\to C_k$) and $((n-1-k)+2)$-gon ($\to C_{(n-1)-k}$)

All choices of k and of the two smaller triangulations are possible, so

$$C_n = \sum_{k=0}^{n-1} C_k \, C_{(n-1)-k}, \text{ as claimed.}$$

$\blacksquare$

Okay, but what's the connection to g.f.'s?...

Algebra says that if $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$

then $A(x)B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.$

So the fund. recurrence says something very nice about the Catalan number g.f.:

$$C(x) = \sum_{n=0}^{\infty} C_n x^n$$

namely,

$$C(x)C(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$$

(fund. rec.) $= \sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=1}^{\infty} C_n x^{n-1}$

$$= \frac{1}{x}\left(C(x) - 1 \right)$$

i.e., $x\,C(x)^2 - C(x) + 1 = 0$

$\Rightarrow C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$ by quad. form.

Remember,

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

$$\int (1-4x)^{-1/2} = \text{const.} + \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1}$$

$$\parallel$$

$$-\frac{1}{2}(1-4x)^{1/2} \overset{x=0}{\rightsquigarrow} \text{const.} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{2}\sqrt{1-4x} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1}$$

$$\Rightarrow \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$$

$$\parallel$$

$$\sum_{n=0}^{\infty} C_n x^n$$

(Since these coeff's are ≥ 0, shows we should take $-$ in $\pm$)

$$\Rightarrow \boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

e.g. $C_4 = \frac{1}{5}\binom{8}{4} = \frac{1}{5} \cdot 70 = 14$

$=$ # triang. of hexagon

So with generating functions we were able easily to find an explicit formula for Catalan numbers.

There are other ways to prove the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$

(can you find a bijective proof???)

but... this proof using g.f.'s is probably the "easiest."

Shows power of generating functions!

Now let's take a break...

And when we come back we can work in breakout groups on the worksheet, which shows many more counting problems where the answer is the Catalan #'s!