Math 4707: Integer partitions

2/15
Not in LPV

Reminders: • HW#1 has been graded, visible on Canvas.
If you have questions/issues w/grading, email me.
• HW#2 is due this Wed., 2/17.
• Midterm #1 will be posted soon, is due Wed. of next week, 2/24.

We're almost done with enumerative combinatorics, our first major topic. Next class we will start with graph theory, the next major topic. Today we will discuss one last popular topic in enumerative combinatorics: the study of (integer) partitions.

Compositions

Before we discuss partitions, let's talk about (integer) compositions, which are closely related.

Defn A composition of n into k parts is a way of writing n as a sum of k positive integers:

$$n = a_1 + a_2 + \cdots + a_k.$$

e.g. $13 = 5+1+5+2$ is a composition of 13 into 4 parts.

$13 = 2+1+5+5$ is a different one.

order matters!

Q: How many comp. of n into k parts are there?

If we didn't have the requirement that $a_i \geq 1$, i.e., if we allowed $a_i = 0$ ('weak compositions') then this is the same as the 'giving n pennies to k children' problem, which had the formula

← a.k.a. bagel problem

$$\binom{n+k-1}{k-1} \quad \text{via stars + bars.}$$

For requirement $a_i \geq 1$, easy fix: first give 1 penny to each kid, then distribute $n-k$ remaining pennies in

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1} \text{ ways.}$$

$\Rightarrow$ Thm # compositions of n into k parts

$$= \binom{n-1}{k-1}$$

Cor # comp. of n into any number of parts

$$= 2^{n-1}$$

pf: Do sum $\sum_{k=1}^{n}\binom{n-1}{k-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$ ✓ ∎

Note: Another direct way to see 2^{n-1} via stars + bars is:

$* \quad * \quad * \quad * \quad *$

bar here? Y/N?

$n=5$

e.g. $* \mid * \; * \; * \mid * = 1+3+1$

Y N N Y

Partitions

A partition of n is like a composition, except that order doesn't matter:

$1+5+5+2$ same as $2+1+5+5$

Def'n A partition of n into k parts is a way of writing $n=\lambda_1+\lambda_2+\cdots+\lambda_k$ w/ positive integers s.t. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

this convention chooses one fixed ordering for any $\lambda_1,\ldots,\lambda_k$.

e.g. We'd write above as $5+5+2+1$

Note often write partition as tuple $\lambda=(\lambda_1,\ldots,\lambda_k)$.

e.g., $\lambda=(5,5,2,1)$.

Let $p(n) := \#$ partitions of n.

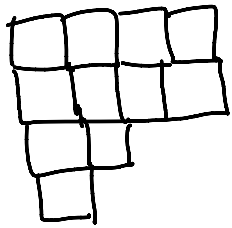
n	partitions	$p(n)$
1	1	1
2	2, 1+1	2
3	3, 21, 111	3
4	4, 31, 22, 211, 1111	5
5	5, 41, 32, 311, 221, 2111, 11111	7
6	???	11

Q: What is $p(n)$?

A: Much, much harder to give formula for $p(n)$ than case of compositions.

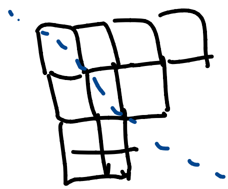
Even if we won't easily be able to count them, let's think a little more about partitions…

∃ a very nice graphical representation of a partition, called its Young diagram:

$4+4+2+1 \iff$  $= \lambda$

We see a new symmetry from Young diagram:

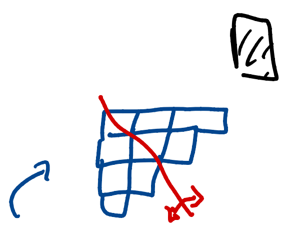
λ partition, its conjugate λ^t has transposed Young diagram:

$4+3+2+2 \iff$  $= \lambda^t$

Prop: # partitions of n into k parts

= # partitions of n w/ largest part $= k$.

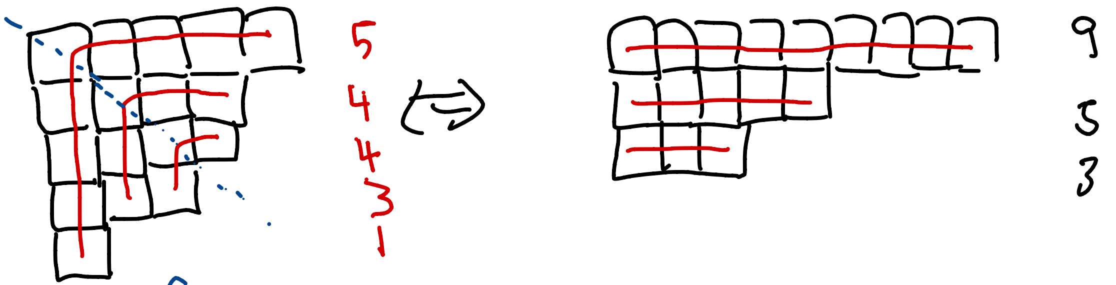
Pf: ???



Can we say anything about self-conjugate partitions (i.e., equal to own conjugate)?

Thm. # Self-conjugate partitions of n $=$ # partitions of n into distinct, odd parts.

Pf: Look at this picture:



any self-conjugate partition can be decomposed into "elbows" like this

We'll see another bijection between two classes of partitions in a moment, but first...

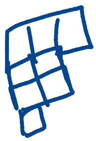
Generating functions

In the past two classes we've seen that generating functions are very powerful tools to help understand sequences, and so what about using them for partitions.

Prop. $\displaystyle\sum_{n\geq 0} p(n)\, q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}$.

Pf. $\displaystyle\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \cdot \frac{1}{1-x^k} \cdots =$ $q = x$

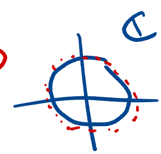
$(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)\cdots(1+x^k+\ldots)$



every choice of a term from each factor creates a unique partition, w/ exponent of n = size. $\square$

Note: We saw before that for rational generating functions $\dfrac{P(x)}{Q(x)} = \sum a_n x^n$, rate of growth of a_n has to do w/ 'singularities', i.e. roots of $Q(x)$.

Singularities of $\displaystyle\prod_{j=1}^{\infty} \frac{1}{1-q^j}$ are much harder to analyze



Still, w/ (a lot of) work, idea leads to

Thm (Hardy–Ramanujan)

$$p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty .$$

What about a more down-to-earth use of g.f.'s for partitions? Can prove more identities ...

Let o_n = # partitions of n into odd parts

d_n = # partitions of n into distinct parts

e.g.

n	odd	distinct	o_n	d_n
1	1	1	1	1
2	11	2	1	1
3	3, 111	3, 21	2	2
4	31, 1111	4, 31	2	2
5	5, 311, 11111	5, 41, 32	3	3

$$O(x) := \sum_{n \geq 0} o_n x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

why?

$$D(x) := \sum_{n \geq 0} d_n x^n = (1+x)(1+x^2)(1+x^3) \cdots$$

$\emptyset$ $\square$ $\emptyset$ $\square\square$ $\emptyset$ $\square\square\square$ etc.

Then,

$$D(x) \cdot \frac{(1-x)}{(1-x)} \cdot \frac{(1-x^2)}{(1-x^2)} \cdot \frac{(1-x^3)}{(1-x^3)} \cdot \frac{(1-x^4)}{(1-x^4)} \cdots$$

$$= (1+x)(1+x^2)(1+x^3) \cdots \frac{(1-x)}{(1-x)} \frac{(1-x^2)}{(1-x^2)} \frac{(1-x^3)}{(1-x^3)} \cdots$$

$$= \frac{(1\cancel{+x})(\cancel{1-x})}{(1-x)} \frac{(1\cancel{+x^2})(1\cancel{-x^2})}{(\cancel{1-x^2})} \frac{(1\cancel{+x^3})(1\cancel{-x^3})}{(1-x^3)} \frac{\cdots}{(\cancel{1-x^4})} \cdots$$

recall $(1-a)(1+a) = (1-a^2)$

$$\cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots = O(x).$$

Cor. $d_n = o_n$ for all n. Pf: extract x^n coefs. ∎

MAGIC!!!

odd parts

distinct parts

Note: $\exists$ bijection $O(n) \to D(n)$ using binary representation!

e.g. $\overset{n}{\overset{\shortparallel}{26}} = 5+5+5+3+3+1+1+1+1+1$ odd parts

$= 3(5) + 2(3) + 5(1)$

binary

$= (2^1+2^0)5 + (2^1)(3) + (2^2+2^0)(1)$

$= 10 + 5 + 6 + 4 + 1$

distinct parts!

think abt. later...

There are many more great theorems about partitions, like:

- Euler's pentagonal number thm,
- Ramanujan's congruences, ...

et cetera.

But now let's take a 5 min. break and when we come back we'll do the worksheet on partitions in breakout groups...