

Math 4707: Intro to graph theory -
walks, circuits, & degrees

2/7
Ch. 7
of LVP

Reminders : • HW #2 due today.

• Midterm #1 has been posted, due in one week on Wed., 2/24.

Today we will start discussing **graph theory**, a topic that we'll explore for most of the rest of the course.

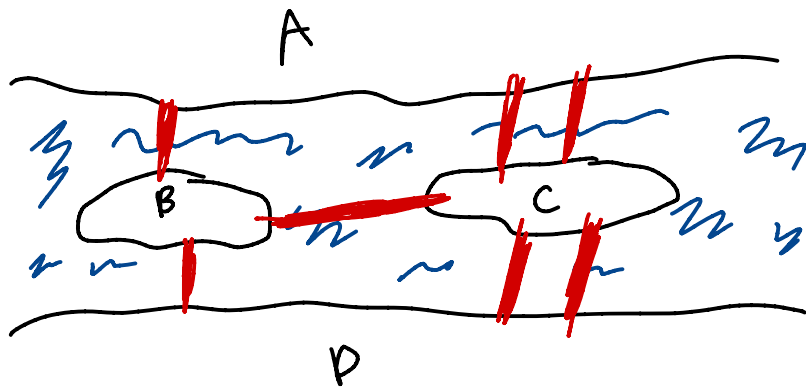
Origin of graph theory: Bridges of Königsberg

Königsberg was a city in 18th century East Prussia whose unique geography led to the birth of graph theory.

It was on a river and had 7 **bridges**.

People wondered whether it was possible to cross every bridge **exactly once** and end

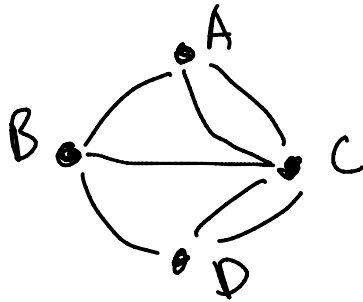
up where you started. Here's how the city looked:



The best mathematician of the time was Leonard **Euler**, and he got interested in this puzzle. He was able to show the answer was **no**, you can't! And in doing so, he developed a lot of the basics of **graph theory**.

Euler realized that the exact shapes of the landmasses in Königsberg were irrelevant to the problem, all that mattered was how they were **connected**. In other words, all the relevant information

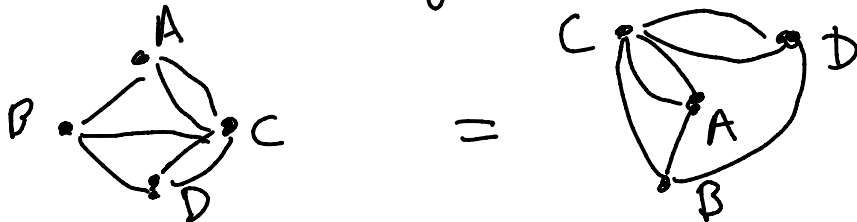
Can be encoded in a **diagram** like this:




A diagram like this is called a **graph**.

Formally, a graph $G = (V, E)$ consists of a set of **vertices** (the **points**, e.g., A, B, C, D) V and a set of **edges** (the **lines** connecting the points) E , where an edge $e \in E$ is an **(unordered) pair** $e = \{u, v\}$ of vertices $u, v \in V$.

We represent graphs via diagrams like the above, but it's important to note that the same graph can be drawn in multiple ways:



As you can imagine, there are many choices we have for the precise definition of a graph:

- Can we have **multiple edges** between the same pair of vertices, or at most one?
- are we allowed to have a **loop**, an edge connecting a vertex to itself? 

Graphs without multiple edges or loops are called **simple** graphs. (Note that the bridges of Königsberg graph is not simple.) We will also discuss another variant of graphs, **directed graphs**, a bit later.

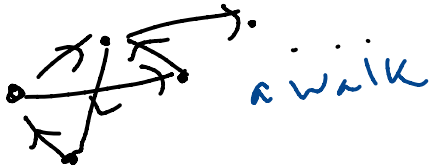
Also worth mentioning that graphs are very **versatile** data structures. In the B.O.K. problem we see graphs used to represent **spatial data** (and we'll see later that **maps** in particular were significant in the development of graph theory), but graphs can be used to represent any **(symmetric) relations**, e.g., **social networks**. ← popular application!

Now back to the B.o.K. problem...

Not only did Euler solve this particular bridge problem, but he gave a solution for any possible arrangement of bridges + landmasses.

Some relevant graph terminology:

- a **Walk** in a graph is a way of walking between vertices along edges; more formally it's a sequence $v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ of vertices v_i and edges e_j such that $e_i = \{v_{i-1}, v_i\}$.



We say that the walk is **from** v_0 **to** v_n .

- a **closed walk** or **circuit** is a walk from a vertex to itself, i.e., ends where it starts.
- an **Eulerian walk** (book calls it **Eulerian trail**) is a walk that uses each edge of graph **exactly once**.
- an **Eulerian circuit** is an Eulerian walk that's a circuit.

So the B.o.k. problem is about the existence of Eulerian circuits in graphs, Euler found a simple exact criterion for the existence of an Eulerian circuit in a graph.

To state this criterion, need just two more pieces of graph theory terminology:

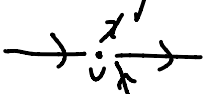
- a graph is **connected** if there is a walk from any vertex to any other vertex. Every graph is the **disjoint union** of its **connected components**:



- for $G=(V,E)$ a graph without loops, and $v \in V$ a vertex, the **degree** of v , denoted $\deg(v)$, is the number of edges containing v . (If G has loops, they count double for degree.)

Thm (Euler) A connected graph $G=(V,E)$ has an Eulerian circuit if and only if every vertex has even degree.

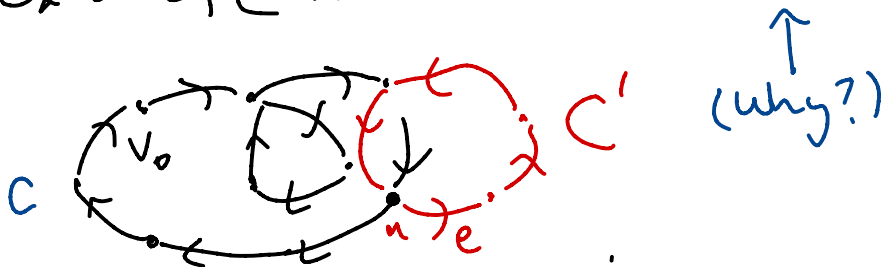
E.g., B.o.k. graph  has degrees 5, 3, 3, 3 \Rightarrow no Eulerian circuit

Pf: (only if direction) In an Eulerian circuit, we walk **into** any vertex v exactly as often as we walk **out of** v .  So certainly v must be incident to an even number of edges.

(If direction) Let's construct an Eulerian circuit. Pick any initial vertex $v_0 \in V$. Start walking from v_0 : whenever we walk into a vertex v , walk out along an edge we haven't yet traversed. Do this as long as we can: by the even degree assumption, the only place we can stop is at v_0 , so we make a circuit C :



If we've used all the edges of G in C then we're done. Otherwise, by connectedness, there must be an edge e leaving a vertex u of C that we didn't use:



Then, as shown above, let's start walking out

from u along that edge, using edges we haven't traversed (including in C). Again, even degrees \Rightarrow we can only get stuck at u , so we get a new circuit C' .

Then we can "join" C and C' : walk from v_0 to u along C , then do C' , then walk from u back to v_0 along C . Repeat this process until we use all edges. \blacksquare

What about if we just want an Eulerian walk?

Thm For $G=(V,E)$ connected, \exists Eulerian walk from s to t w/ $s \neq t \in V$, iff s and t have odd degree and all other vertices have even degree.

Pf: Think about adding edge $s \text{ --- } t$ \blacksquare


Ex. B.s.k. graph doesn't even have Euler. walk!


These Eulerian walk/circuit thms are prototypical graph theory results characterizing existence of structures.

[Aside on history of Königsberg:  \Rightarrow 

Hamiltonian paths

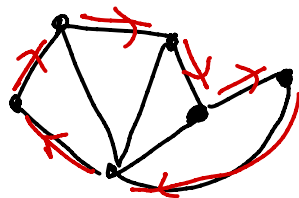
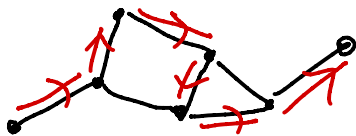
Let's discuss another substructure question:

- a **path** in a graph is a walk that doesn't repeat any vertices: 

- a **cycle** in a graph is a circuit that doesn't repeat vertices, except start=end 

- a **Hamiltonian path** is a path that uses every vertex.

- a **Hamiltonian cycle** is cycle using every vertex:



Q: When does a Hamiltonian path/cycle exist in a graph?

A: Much **harder** to say than for Eulerian walk/circuit; no useful exact criterion (and even hard for a **computer!**)

Note: adding edges to your graph only improves ability to find Hamilton. path/cycle, so there are **sufficient conditions** saying graphs w/ many edges have Hamilton. path/cycles, e.g.:

[show Hamilton's iosian game ...]

More on degrees

We saw w/ Euler's theorem that degrees of vertices can be very important ...

Q: Let $G = (V, E)$ be a graph (w/o loops). What's another expression for $\sum_{v \in V} \deg(v)$?

A: $2 \cdot \#E$, since $\overset{e}{u} \rightarrow v$ contributes 2 degree!

Cor: # vertices w/ odd degree is **even**.

Pf: ???



e.g., at any **party**, an **even** # of people are friends w/ an odd # of people!

Now let's take a 5 min break.
And when we come back,
try to find some walks/
circuits in graphs on the
worksheet in breakout groups.