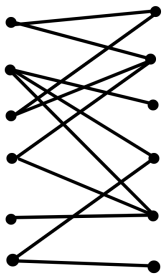
Matchings,
Math 4707, Spring 2021

1. Let G be the following bipartite graph on 12 vertices:



   Find a matching of G with the maximum possible number of edges. How do you *know* that this is the maximum?

2. Let G be a simple bipartite graph with bipartition (X, Y). Suppose that $n = \#X = \#Y$ (so the total number of vertices of G is $2n$).

   (a) If G has a perfect matching, must it be connected?

   (b) What is the fewest number of edges G could have if it has a perfect matching.

   (c) Show that G can have $n^2 - n$ edges but still fail to have a perfect matching.

   (d) Show that if G has at least $n^2 - n + 1$ edges, then it must have a perfect matching.
   Hint: suppose there is a subset $A \subseteq X$ with $a = \#A > \#N_G(A)$, where $N_G(A)$ denotes the neighborhood of A in G. Write an expression (in terms of a and n) for the maximum possible number of edges G could have in that case. Show that your expression cannot be greater than or equal to $n^2 - n + 1$.