Matrices and graphs,
Math 4707, Spring 2021

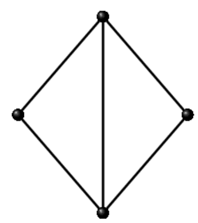
1. Show that the number of closed walks of length n in the complete graph K_3 on three vertices is $2^n + 2 \cdot (-1)^n$ by computing the eigenvalues of its adjacency matrix.

2. Explain how closed walks of length n in K_3 are in bijection with words $w = (w_1, ..., w_{n+1})$ of size $n+1$ in the alphabet $\{a, b, c\}$ for which $w_i \neq w_{i+1}$ for all $i = 1, 2, ..., n$ and with $w_{n+1} = w_1$.

3. Explain how the words from the previous question are in bijection with ways to color the vertices of the cycle graph C_n on n vertices with three different colors so that adjacent vertices don't have the same color. These kind of graph colorings are usually called *proper colorings*, or even just *colorings* for short. Conclude that the number of proper 3-colorings of C_n is $2^n + 2 \cdot (-1)^n$.

4. Can you generalize the above to count proper k-colorings of C_n? **Hint**: find the eigenvalues of the complete graph K_k on k vertices!

5. Find all the spanning trees of:



6. Use the Matrix-Tree Theorem to count the spanning trees of the graph from the previous question.

7. The *complete bipartite graph* $K_{n,m}$ is the graph with vertex set $X \cup Y$ where $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$, and with edges $\{x_i, y_j\}$ for all $1 \leq i \leq n, 1 \leq j \leq m$ (but with no edges between the x's, or between the y's). Use the Matrix-Tree Theorem to show that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$. **Hint**: you can use the fact from linear algebra that for a *block matrix* M of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have $\det(M) = \det(A - BD^{-1}C)\det(D)$ as long as D is invertible. (Observe that this is a generalization of the 2×2 determinant formula $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.)