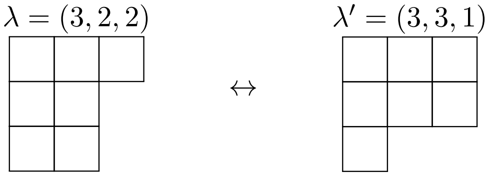
Partitions,
Math 4707, Spring 2021

An *(integer) partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots)$ is a sequence of weakly decreasing nonnegative integers which is eventually zero. The *size* of λ is $|\lambda| := \lambda_1 + \lambda_2 + \ldots.$ The *largest part* of λ is λ_1, and the *length* $\ell(\lambda)$ of λ is the number of (nonzero) parts of λ, i.e., $\ell(\lambda) := \max\{i \colon \lambda_i \neq 0\}$.

The *conjugate* λ' of λ is the partition whose Young diagram is obtained from that of λ by reflecting across the main diagonal:

$\lambda = (3, 2, 2)$ $\qquad\qquad$ $\lambda' = (3, 3, 1)$

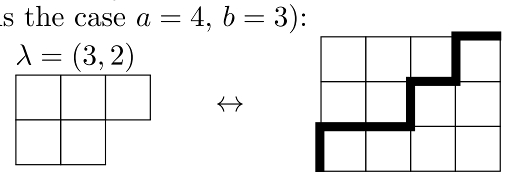


1. Show that the generating function for partitions with largest part at most k is $\sum_{\lambda\colon \lambda_1 \leq k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1}{1-q^i}$.

2. Show that the generating function for partitions with length at most k is $\sum_{\lambda\colon \ell(\lambda) \leq k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1}{1-q^i}$ (hint: use the conjugate).

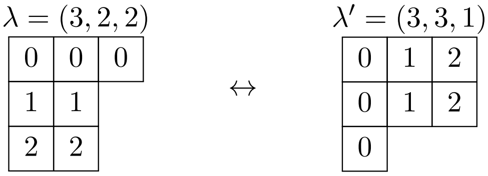
3. Show that the number of partitions λ with largest part at most a and length at most b is $\binom{a+b}{b}$. Hint: think about the following picture (which explains the case $a = 4$, $b = 3$):

   $\lambda = (3, 2)$

4. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ its conjugate. Show that $\sum_{i \geq 1} (i-1) \cdot \lambda_i = \sum_{i \geq 1} \binom{\lambda'_i}{2}$. Hint: think about this picture:

   $\lambda = (3, 2, 2)$ $\qquad\qquad$ $\lambda' = (3, 3, 1)$

5. Describe the partition λ of size $|\lambda| = n$ which maximizes $\min(\lambda_1, \ell(\lambda))$.