Pick's theorem,
Math 4707, Spring 2021

Let P be a lattice polygon with vertices $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{Z}^2$ in clockwise cyclic order. For k a positive integer, we use kP to denote the *kth dilate* of P, which is the polygon with vertices $(kx_1, ky_1), \ldots, (kx_n, ky_n)$. Intuitively kP is P "stretched" by a factor of k.

1. Let P be a lattice polygon. Explain why $b(kP) = kb(P)$, i.e., the number of boundary lattice points of kP is k times the number of boundary lattice points of P.

2. Conclude from part (1) and Pick's theorem, that the total number of lattice points in kP is

$$\#(kP \cap \mathbb{Z}^2) = \text{area}(P) \cdot k^2 + (b(P)/2) \cdot k + 1,$$

   which is a **polynomial** in k!

3. Let P be the *unit square*, i.e., the lattice quadrilateral with vertices $(1, 1), (1, 0), (0, 0), (0, 1)$. Compute the polynomial $\#(kP \cap \mathbb{Z}^2)$ from part (2). Do you recognize this sequence of numbers?

4. Let P be the *standard triangle*, i.e., the lattice triangle with vertices $(1, 0), (0, 0), (0, 1)$. Compute the polynomial $\#(kP \cap \mathbb{Z}^2)$ from part (2). Do you recognize this sequence of numbers?

Remark: The higher-dimensional version of a convex polygon is called a *convex polyhedron*: a *convex polyhedron* P in $\mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. Polygons are built out of "flat" things: vertices and sides (a.k.a. edges); polyhedra are also built out of flat things: vertices, edges, faces, etc. You might be aware of the *Platonic solids* in $\mathbb{R}^3$.

We say P is a *lattice polyhedron* if it is the convex hull of finitely many points in $\mathbb{Z}^d$. For P a lattice polyhedron, a version of (2) holds:

$$\#(kP \cap \mathbb{Z}^d) = a_d \cdot k^d + a_{d-1} \cdot k^{d-1} + \cdots + a_1 \cdot k + a_0,$$

i.e., the number of lattice points of kP is given by a *polynomial* in P called the *Ehrhart polynomial* of P. A whole area of combinatorics, called *Ehrhart theory*, studies these polynomials.