Active learning exercise on
Stirling numbers of the second kind
Vic Reiner, Math 4707, Monday Oct. 10, 2011

For positive integers n, k with $1 \leq k \leq n$, the *Stirling number of the second kind* $S(n,k)$ is the number of ways to partition a set of n labelled elements, such as $\{1, 2, \ldots, n\}$, into k unlabelled nonempty blocks.

For example, listing partitions as lists of blocks with dashes between them (and suppressing set brackets and commas), one has

$$\begin{aligned} S(4,2) = 7 = |\{&123-4, \quad 124-3, \quad ,134-2, \quad 234-1, \\ &12-34, \quad 13-24, \quad 14-23\}|, \\ S(3,1) = 1 = &|\{123\}|, \\ S(3,2) = 3 = &|\{12-3, \quad 13-2, \quad 23-1\}|, \\ S(3,3) = 1 = &|\{1-2-3\}|. \end{aligned}$$

One can create a triangle of the numbers $S(n,k)$ similar to Pascal's triangle for the binomial coefficients $\binom{n}{k}$, starting like this:

$n \setminus k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	?	7	?	?	
5	?	?	?	?	?

1. Compute the remaining entries $S(4,1), S(4,3), S(4,4)$. in the $n = 4$ row of the above table.

2. Explain why for all n, one has
(a) $S(n,1) = 1 = S(n,n)$,
(b) $S(n, n-1) = \binom{n}{2}$,
(c) $S(n,2) = \frac{2^n - 2}{2} = 2^{n-1} - 1$.
(d) Use parts (a),(b),(c) to check your answers for the $n = 4$ row of the table, and use them to fill in almost all of the entries in the $n = 5$ row, except $S(5,3)$.
(e) Explain why $S(5,3) = \binom{5}{3,2} + \frac{1}{2}\binom{5}{2,2,1}$, and use this to fill it in.

3. Decreeing that $S(n,k) = 0$ unless $1 \leq k \leq n$, explain why for $n \geq 2$,

$$S(n,k) = S(n-1, k-1) + kS(n-1, k).$$

Use this to re-compute the $S(5,3)$ value from exercise 2(e).

For each fixed $k \geq 1$, define an *(ordinary) generating function* for the sequence $S(k,k), S(k+1,k), S(k+2,k), \ldots$

$$\begin{aligned} f_k(x) &:= \sum_{n=k}^{\infty} S(n,k)x^n \\ &= S(k,k)x^k + S(k+1,k)x^{k+1} + S(k+2,k)x^{k+2} + \cdots \end{aligned}$$

4. Using the formulas for $S(n,1), S(n,2)$ in exercise 2(a,c), show that
(a)

$$\begin{aligned} f_1(x) &= S(1,1)x^1 + S(2,1)x^2 + S(3,1)x^3 + \cdots \\ &= \frac{x}{1-x}, \end{aligned}$$

(b)

$$\begin{aligned} f_2(x) &= S(2,2)x^2 + S(3,2)x^3 + S(4,2)x^4 + \cdots \\ &= \frac{x^2}{(1-x)(1-2x)}. \end{aligned}$$

5. Show using the recurrence in exercise 3 that for $k \geq 2$,

$$f_k(x) = \frac{x}{1-kx} \cdot f_{k-1}(x)$$

and use this to deduce that

$$f_k(x) = \frac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}.$$

In order to derive a summation formula for $S(n,k)$, it is helpful to consider a closely related number $\hat{S}(n,k)$, defined to be the number of ways to partition $\{1,2,\ldots,n\}$ into k *labelled* blocks.

Alternatively, $\hat{S}(n,k)$ is the number of ways to places balls labelled $1,2,\ldots,n$ into k boxes labelled box 1, box 2, …, box k, in such a way that no box ends up empty.

For example, using dashes to separate balls in box i from box $i+1$,

$$\hat{S}(4,2) \quad = 14 \quad = \left| \left\{ \begin{array}{llll} 123-4, & 124-3, & 134-2, & 234-1, \\ 4-123, & 3-124, & 2-134, & 1-234, \\ 12-34, & 13-24, & 14-23, & \\ 34-12, & 24-13, & 23-14 & \end{array} \right\} \right|$$

$$\hat{S}(3,3) \quad = 6 \quad = \left| \left\{ \begin{array}{lll} 1-2-3, & 1-3-2, & 2-1-3, \\ 2-3-1, & 3-1-2, & 3-2-1 \end{array} \right\} \right| .$$

6. Describe a simple relation between the numbers $\hat{S}(n,k)$ and $S(n,k)$.

7. Write down as a function of n and k how many ways there are to put balls labelled $\{1,2,\ldots,n\}$ into the k labelled boxes if ...
(a) there are *no restrictions* about boxes being empty or not,
(b) one insists that box i *must* be empty,
(c) one insists that boxes i and j *must both* be empty,
(d) one insists that boxes $i_1, i_2, \ldots, i_j$ *must all* be empty.

8. We claim that one has the following summation formula for $\hat{S}(n,k)$:

$$\hat{S}(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \tag{1}$$

(a) Check that the formula (1) gives the correct answer for $\hat{S}(4,2)$.
(b) What formula for $\hat{S}(n,2)$ does (1) give when you plug in $k=2$?
(c) Prove formula (1) (Hint: Why is problem 7 relevant?)
(d) Use (1) and Exercise 6 to derive a summation formula for $S(n,k)$.