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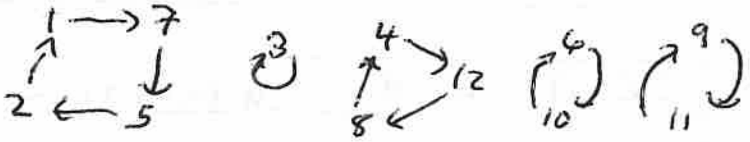
Permutations and cycles (Stanley §1.3)

Recall $\mathfrak{S}_n$ = symmetric group on n letters
= permutations of $[n]$

Notations:

- two-line $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 1 & 3 & 12 & 2 & 10 & 5 & 4 & 11 & 6 & 9 & 8 \end{pmatrix}$
- one-line $\sigma = (7, 1, 3, 12, 2, 10, 5, 4, 11, 6, 9, 8)$
- "directed graph" → functional digraph: $\sigma =$



- cycle notation: $\sigma = (1\,7\,5\,2)\ (3)\ (4\,12\,8)\ (6\,10)\ (9\,11)$

$= (8\,4\,12)\ (10\,6)\ (5\,2\,7)\ (3)\ (11\,9)$

$=$ etc $= (\underline{3})(\underline{7}\,5\,2\,1)(\underline{10}\,6)(\underline{11}\,9)(\underline{12}\,8\,4)$

Standard form:
- each cycle has its biggest element first
- cycles appear w/ biggest elements increasing left-to-right.

Q: How many $\sigma \in S_n$ of cycle type $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$
$= 1^{c_1} 2^{c_2} 3^{c_3} \dots$

e.g. $n = 4$

$\lambda = 1^4 =$  $(a)(b)(c)(d)$ 1

$2^1 1^2 =$  $(ab)(c)(d)$ $\binom{4}{2} = 6$

$2^2 =$  $(ab)(cd)$ $\binom{4}{2}/2 = 6/2 = 3$

$3^1 1^1 =$  $(abc)(d)$ $2! \cdot \binom{4}{3} = 2 \cdot 4 = 8$

$4^1 =$  $(abcd)$ $3! \cdot \binom{4}{4} = 6$

multiplicity notation:
e.g. $\lambda = (5, 5, 5, 3, 2, 2, 2, 2, 1, 1)$
$= 1^2 2^4 3^1 4^0 5^3$
$c_1 = 2,\ c_2 = 4,\ c_3 = 1,\ c_4 = 0,\ c_5 = 3$

Prop: There are $\dfrac{n!}{1^{c_1} c_1! \, 2^{c_2} c_2! \, 3^{c_3} c_3! \dots}$ perms in $\mathfrak{S}_n$ of cycle type $\lambda = 1^{c_1} 2^{c_2} 3^{c_3} \dots$

Pf of prop: Recall that $\mathfrak{S}_n$ acts on the set of perms with cycle type λ transitively, by conjugation:

$$\text{e.g. } \underbrace{\begin{pmatrix} 1&2&3&4&5&6&7 \\ a&b&c&d&e&f&g \end{pmatrix}}_{\sigma} (1234)(567) \underbrace{\begin{pmatrix} a&b&c&d&e&f&g \\ 1&2&3&4&5&6&7 \end{pmatrix}}_{\sigma^{-1}}$$

$$= (abcd)(efg)$$

So the # of such perms = size of the orbit

$$\text{orbit-stabilizer} \longrightarrow \; = \frac{|S_n|}{|Z_{S_n}(\sigma_\lambda)|} \quad \text{if } \sigma_\lambda \text{ is a perm of cycle type } \lambda,$$

where $Z_{S_n}(\sigma) := \{\tau \in S_n : \tau\sigma_\lambda = \sigma_\lambda\tau\}$ is the centralizer of σ_λ in $\mathfrak{S}_n$.
i.e., $\tau\sigma_\lambda\tau^{-1} = \sigma_\lambda$

Who centralizes $\sigma_\lambda = \underbrace{(a)(b)\cdots}_{c_1 \text{ 1-cycles}} \underbrace{(cd)(ef)\cdots}_{c_2 \text{ 2-cycles}} \cdots \cdots$?

- Products of powers of each cycle: there are $1^{c_1}2^{c_2}3^{c_3}\cdots$ of these
- Perms that swap two cycles of same size, preserving cyclic order and biggest element: there are $c_1!\,c_2!\,c_3!$ of these

products of these things = $1^{c_1}c_1!\,2^{c_2}c_2!\cdots$ many

e.g. $(1234)(567)(8\,9\,10) = \sigma_\lambda$

is centralized by $\tau = \underbrace{(4321)}_{(1234)^3} \underbrace{(59)(6\,10)(78)}_{\text{swap } (567) + (8\,9\,10)}$

Thus $|\text{orbit}| = \dfrac{n!}{\prod_{j\geq 1} j^{c_j} c_j!}$, as claimed. ☒

NOTE: Stanley presents different (but equivalent) proof by considering standard forms of perms σ_λ.

DEF'N: For any subgroup G of $\mathfrak{S}_n$, define its cycle index (indicator) polynomial to be

$$Z_G(t_1,t_2,\ldots) := \frac{1}{|G|}\sum_{\sigma\in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots \in \mathbb{C}[t_1,t_2,\ldots,t_n]$$

where $c_i(\sigma) :=$ # cycles in σ of size i.

Cor (Touchard): The cycle indicators $Z_{\mathfrak{S}_n}$ have g.f.

$$\sum_{n=0}^{\infty} Z_{\mathfrak{S}_n}(t)\, x^n = e^{t_1\frac{x}{1}+t_2\frac{x^2}{2}+t_3\frac{x^3}{3}+\cdots} = e^{\sum_{j\geq 1} t_j \frac{x^j}{j}}$$

Proof (direct but mysterious..., we'll see better pf later)

$$e^{\frac{t_1 x^1}{1}+\frac{t_2 x^2}{2}+\cdots} = e^{\frac{t_1x^1}{1}} e^{\frac{t_2x^2}{2}}\cdots$$

$$= \left(\sum_{c_1\geq 0}\frac{\left(\frac{t_1x^1}{1}\right)^{c_1}}{c_1!}\right)\left(\sum_{c_2\geq 0}\frac{\left(\frac{t_2x^2}{2}\right)^{c_2}}{c_2!}\right)\cdots$$

$$= \sum_{(c_1,c_2,\ldots)} x^{1\cdot c_1+2\cdot c_2+\cdots}\, \frac{t_1^{c_1}t_2^{c_2}\cdots}{1^{c_1}c_1!\,2^{c_2}c_2!\cdots}$$

$$= \sum_{n\geq 0} x^n \frac{1}{n!}\sum_{\substack{(c_1,c_2,\ldots)\\ \sum j c_j = n}} \boxed{\frac{n!}{1^{c_1}c_1!\,2^{c_2}c_2!\cdots}}\; t_1^{c_1}t_2^{c_2}\cdots$$

$= \#\{\sigma\in\mathfrak{S}_n : \sigma \text{ has } c_j\ j\text{-cycles}\}$

$$= \sum_{n\geq 0} x^n \underbrace{\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)}t_2^{c_2(\sigma)}\cdots}_{Z_{\mathfrak{S}_n}(t)}$$

▨

Touchard's thm has many consequences...
Let's see a few now...

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① <u>DEF'N</u> $$\sum_{k=1}^{n} \underbrace{c(n,k)}_{\text{(signless) Stirling number of 1st kind}} t^k := \sum_{\sigma\in\mathfrak{S}_n} t^{\#\text{cycles}(\sigma)}$$

i.e., $c(n,k) := \#\{\sigma\in S_n : \sigma \text{ has } k \text{ cycles}\}$.

Cor (to Touchard) $\sum_{k=1}^{n} c(n,k)\, t^k = t(t+1)(t+2)\cdots(t+(n-1))$

Pf: Set $t_1=t_2=\cdots=t$ in

$$\sum_{n\geq 0}\frac{x^n}{n!}\sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)}\, t_2^{c_2(\sigma)}\cdots = e^{t_1\frac{x^1}{1}+t_2\frac{x^2}{2}+t_3\frac{x^3}{3}+\cdots}$$

$$\Downarrow$$

$$\sum\frac{x^n}{n!}\underbrace{\sum_{\sigma\in\mathfrak{S}_n} t^{\#\text{cycles}(\sigma)}}_{\sum_{k=1}^{n} c(n,k)t^k} = e^{t\left(\frac{x^1}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)}$$

$$= e^{t(-\log(1-x))}$$

$$= (1-x)^{-t}$$

$$= \sum_{n\geq 0}\binom{-t}{n}(-x)^n$$

$$= \sum_{n\geq 0}\underbrace{\binom{t+n-1}{n}}_{=\frac{t(t+1)(t+2)\cdots(t+(n-1))}{n!}} x^n$$

Pf follows by comparing coeff's of $\frac{x^n}{n!}$ ▨

Aside on posets: We've mentioned posets, but let's formally introduce them...

A (finite) poset $(P,\leq)$ is a finite set P together with a partial order $\leq$ on pairs of elem's of P:

- (reflexive) $x\leq x \quad \forall x\in P$
- (antisymmetric) $x\leq y$ and $y\leq x \Rightarrow x=y \quad \forall x,y\in P$
- (transitive) $x\leq y$ and $y\leq z \Rightarrow x\leq z \quad \forall x,y,z\in P$.

Say y covers x in P, denoted $x\lessdot y$, if $x<y$ and there is no z in P with $x<z<y$.

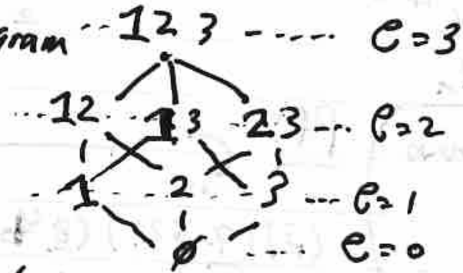
drawn in the plane!

The Hasse diagram of P is the graph w/ elements P and with an edge $x \to y$, and x below y, iff $x \lessdot y$.

E.g.,

recall $B_3 =$ (subsets of $\{1,2,3\}$, $\leq$) Boolean lattice

Hasse diagram $\rightsquigarrow$



(just take transitive closure)

A finite poset is determined by its Hasse diagram.

DEF'N A chain in P is a totally ordered subset $p_1 < p_2 < \cdots < p_m$ of elem's in P. Say P is graded if all maximal chains have the same length. In this case, length $= m-1$

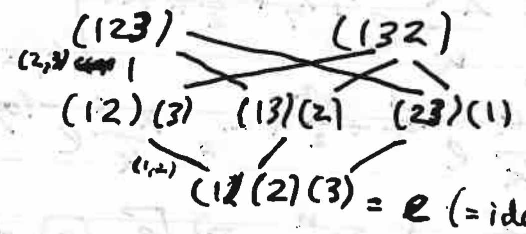
$\exists$ unique rank function $\rho : P \to \{0,1,2,\cdots\}$ s.t.
- $\rho(x) = 0$ if x is minimal in P
- $\rho(y) = \rho(x)+1$ if $x \lessdot y$.

We saw that the binomial coeff's $\binom{n}{k}$ give the rank sizes of a (nice) (graded) poset: the Boolean lattice B_n.

Same is true of Stirling #'s of 1st kind $c(n,k)$:

e.g., $n=3$

$t(t+1)(t+2)$
$= t^3 + 3t^2 + 2t$



$= e$ (= identity)

$c(3,1) = 2$

$c(3,2) = 3$

$c(3,3) = 1$

Partial order on $\mathfrak{S}_n$ called absolute order $(\mathfrak{S}_n, \leq_{abs})$ defined by $\sigma \lessdot_{abs} \tau$ if $\tau = \sigma \cdot (i,j)$ for some i,j and # cycles $(\tau) =$ # cycles $(\sigma) - 1$

(as opposed to # cycles $(\tau) =$ # cycles $(\sigma)+1$, the other possibility for $\tau = \sigma \cdot (i,j)$)

NOTE: Hasse diagram of $(\mathfrak{S}_n, \leq_{abs})$ = Cayley graph of $\mathfrak{S}_n$ w/ transpositions as generating set

Prop!

ther **rmk**: The map $\mathfrak{S}_n \to \mathfrak{S}_n$, $\sigma \mapsto \hat{\sigma} =$ put σ in standard form and erase parentheses

is a bijection, w/ $\#\text{cycles}(\sigma) = \#$ L-to-R Maxima in $\hat{\sigma}$.

Hence $$\sum_{\sigma\in\mathfrak{S}_n} t^{\#\text{L-to-R maxima}(\sigma)} = t(t+1)\cdots(t+(n-1))$$

e.g. $n=3$

σ	#L-to-R max
$\underline{1}\,\underline{2}\,\underline{3}$	3
$\underline{1}\,\underline{3}\,2$	2
$\underline{2}\,1\,\underline{3}$	2
$\underline{2}\,\underline{3}\,1$	2
$\underline{3}\,1\,2$	1
$\underline{3}\,2\,1$	1

Pf: $$\sigma \longmapsto \hat{\sigma}$$

$$(\underline{3})(\underline{7}521)(\underline{8}46) \qquad 3 \vdots 7521 \vdots 846$$

is reversible; just put (before L-to-R maxima, and put) right before the (and at the end. $\blacksquare$

② (Cor of Touchard) Can compute $E_k(n) :=$ expected # of k-cycles in uniformly random $\sigma\in\mathfrak{S}_n$.

$$E_k(n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} c_k(\sigma) = \frac{1}{n!}\left[\partial/\partial t_k \sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_1=t_2=\cdots=1}$$

So $$\sum_{n\geq 0} E_k(n)\, x^n = \left[\partial/\partial t_k \sum_{n\geq 0}\frac{x^n}{n!}\sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_1=t_2=\cdots=1}$$

$$= \left[\partial/\partial t_k\, e^{t_1\frac{x}{1} + t_2\frac{x^2}{2} + t_3\frac{x^3}{3} + \cdots}\right]_{t_i=1}$$

$$= \left[\frac{x^k}{k} e^{t_1\frac{x^1}{1} + \frac{x^2}{2} + \cdots}\right]_{t_i=1}$$

$$= \frac{x^k}{k} e^{\frac{x^1}{1} + \frac{x^2}{2} + \cdots} = \frac{x^k}{k} e^{-\log(1-x)} = \frac{x^k}{(1-x)}$$

$$= \sum_{n\geq k}\frac{1}{k}x^n \Rightarrow E_k(n) = \begin{cases} 1/k & \text{if } n\geq k, \\ 0 & \text{otherwise.}\end{cases}$$ $\blacksquare$

Note: $E_k(n)$ eventually constant in n. In fact, can show $E_k(n)$ converges (as $n\to\infty$) to a Poisson random variable w/ expectation $\lambda = 1/k$.

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③ (Cor of Touchard) There are special classes of perms defined by restrictions on their cycle sizes, so all have nice gen. fn's.

e.g. no large cycles

$\sigma \in \mathfrak{S}_n$ is an involution $(\sigma^2 = e)$

$\iff \sigma$ has only 1- and 2-cycles (i.e., $\sigma = (ab)(cd)\cdots(x)(y)(z)$)

Hence $\displaystyle\sum_{n\geq 0} \frac{x^n}{n!} \#\left\{\begin{matrix}\text{involutions}\\ \text{in } \mathfrak{S}_n\end{matrix}\right\} = \left[e^{t_1\frac{x}{1}+t_2\frac{x^2}{2}+t_3\frac{x^3}{3}+\cdots}\right]_{\substack{t_1=t_2=1\\ t_3=t_4=\cdots=0}}$

$= e^{x+\frac{x^2}{2}}$

or even $\displaystyle\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\substack{\text{involutions}\\ \sigma\in\mathfrak{S}_n}} t^{\#1\text{-cycles}(\sigma)} = e^{tx+\frac{x^2}{2}}$ similarly, etc.

What about no small cycles?

DEF'N A derangement $\sigma \in \mathfrak{S}_n$ is a permutation w/ no fixed points, equivalently, w/ $c_1(\sigma) = 0$.

Q: (Derangement/Hat-check problem): $n=$100 people check their hats; the attendant gives people back their hats completely randomly; what is the probability that no person gets their own hat back?

i.e., what is $\frac{d_n}{n!}$, where $d_n := \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ derangement}\}$

$$\sum_{n\geq 0} \frac{x^n}{n!} d_n = \left[e^{t_1\frac{x}{1}+t_2\frac{x^2}{2}+\cdots}\right]_{t_1=0,\ t_2=t_3=\cdots=1}$$

$$= e^{\frac{x^2}{2}+\frac{x^3}{3}+\cdots}$$

$$= e^{-\log(1-x)-\frac{x^1}{1}} = \boxed{\frac{e^{-x}}{1-x}}$$

But $$\frac{e^{-x}}{1-x} = (1+x+x^2+\cdots)\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)$$

$$= \sum_{n\geq 0} x^n \Big(\underbrace{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}}_{\shortparallel}\Big)$$

$$\frac{d_n}{n!} \xrightarrow[\text{quickly}]{\text{converges}} e^{-1} = \frac{1}{e}$$

Consistent w/ $c_1(\sigma) \to$ Poisson w/ mean $\lambda = 1$

A digression on Pólya theory

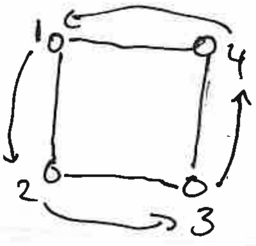
Recall that we defined the cycle index $Z_G(t) := \frac{1}{|G|}\sum_{\sigma\in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots$ for any permutation group $G \subseteq \mathfrak{S}_n$.

Why? Pólya theory! = counts G-orbits of colorings of a finite set X (where $G \circlearrowright X$) with k colors $a_1, a_2, \ldots, a_k$, and more generally ~~to~~ studies the pattern inventory

$$\sum_{\substack{G\text{ orbits }\mathcal{O}\\ \text{of } k\text{-colorings of } X}} a_1^{\#\text{ times color 1 is used}}\, a_2^{\#\text{ times color 2 is used}} \cdots$$

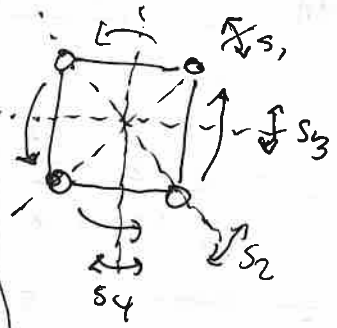
Examples:

$G = C_4$ = cyclic group of order 4 $= \langle (1234) \rangle$



acting on X = vertices of square and 3-colorings via colors $\{a,b,c\}$ ($\overset{\shortparallel}{k}$)

$G = D_8$ = dihedral gp. of order 8



with X = square and colors $\{a,b,c\}$ as before

(Note: Colorings
n-gon up to cyclic symmetry & up to dihedral symmetry)
= "necklaces" = "bracelets"

Pattern inventory for C_4:

$\begin{matrix} a & - & a \\ | & & | \\ a & - & a \end{matrix}$ ${}^4+b^4+c^4$ | $\begin{matrix} a & - & a \\ | & & | \\ a & - & b \end{matrix}$ $+a^3b+a^3c+ab^3+ac^3+bc^3+b^3c$ | $\begin{matrix} a & - & a \\ | & & | \\ b & - & b \end{matrix}$ $\begin{matrix} a & - & b \\ | & & | \\ b & - & a \end{matrix}$ $+2a^2b^2+2a^2c^2+2b^2c^2$ | $\begin{matrix} a & - & a \\ | & & | \\ c & - & b \end{matrix}$ $\begin{matrix} a & - & a \\ | & & | \\ b & - & c \end{matrix}$ $\begin{matrix} a & - & b \\ | & & | \\ c & - & a \end{matrix} = \cancel{\begin{matrix} a & - & c \\ | & & | \\ b & - & a \end{matrix}}$ $+3a^2bc+3ab^2c+3abc^2$

Pattern inventory for D_8:

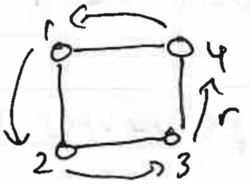
$\begin{matrix} a & - & a \\ | & & | \\ a & - & a \end{matrix}$ | $\begin{matrix} a & - & a \\ | & & | \\ a & - & b \end{matrix}$ | $\begin{matrix} a & - & a \\ | & & | \\ b & - & b \end{matrix}$ $\begin{matrix} a & - & b \\ | & & | \\ b & - & a \end{matrix}$ | $\begin{matrix} a & - & a \\ | & & | \\ c & - & b \end{matrix} = \cancel{\begin{matrix} a & - & a \\ | & & | \\ b & - & c \end{matrix}}$ $\begin{matrix} a & - & b \\ | & & | \\ c & - & a \end{matrix}$

$\underbrace{a^4+b^4+c^4+\cdots}_{\text{same as for } C_4}$ $+2a^2bc+2ab^2c+2abc^2$

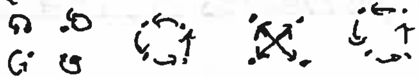
Thm (Pólya) The # of G-orbits of k-colorings of X is $\frac{1}{|G|}\sum_{\sigma\in G} k^{\#\text{cycles}(\sigma)}$

and the pattern inventory is $\left[\underbrace{\frac{1}{|G|}\sum_{\sigma\in G} t_1^{c_1(\sigma)}t_2^{c_2(\sigma)}\cdots}_{Z_G(t)}\right]_{t_j=a_1^j+a_2^j+\dots+a_k^j}$

// EXAMPLES ① $G=C_4=$



$= \{\overset{(1)(2)(3)(4)}{\overset{\shortparallel}{e}}, \overset{(1234)}{\overset{\shortparallel}{r}}, \overset{(13)(24)}{\overset{\shortparallel}{r^2}}, \overset{(1432)}{\overset{\shortparallel}{r^3}}\}$



$\underset{(\text{Polya})}{Z_G(t)} = \frac{1}{4}(t_1^4+t_4^1+t_2^2+t_4^1) = \frac{1}{4}(t_1^4+t_2^2+2t_4)$

$t_j=a^j+b^j+c^j$

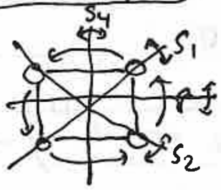
Pattern inventory $= \frac{1}{4}((a+b+c)^4+(a^2+b^2+c^2)^2+2(a^4+b^4+c^4))$

$[a^2b^1c^1]$

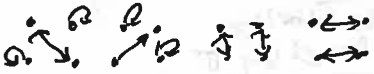
$\boxed{\#\ \overset{3}{\text{colorings}} \text{ w/ } 2a\text{'s}, 1b, 1c} = \frac{1}{4}\left(\binom{4}{2,1,1}+0+0\right) = \frac{1}{4}\frac{4!}{2!1!1!} = \overset{\shortparallel}{3}$

$\# \left\{\begin{matrix} a & - & a \\ | & & | \\ c & - & b \end{matrix}, \begin{matrix} a & - & a \\ | & & | \\ b & - & c \end{matrix}, \begin{matrix} a & - & b \\ | & & | \\ c & - & a \end{matrix}\right\}$

② $G=D_8=$



$=\{e, r, r^2, r^3, s_1, s_2, s_3, s_4\}$



$Z_G(t) = \frac{1}{8}(t_1^4+t_4+t_2^2+t_4+t_2t_1^2+t_2t_1^2+t_2^2+t_2^2)$

$= \frac{1}{8}(t_1^4+3t_2^2+2t_4+2t_2^1t_1^2)$

$t_j=a^j+b^j+c^j$

Pattern inventory $= \frac{1}{8}((a+b+c)^4+3(a^2+b^2+c^2)^2+2(a^4+b^4+c^4)+2(a^2+b^2+c^2)(a+b+c)^2)$

$[a^2b^1c^1]$

$\# \left\{\begin{matrix} a & - & a \\ | & & | \\ c & - & b \end{matrix}, \begin{matrix} a & - & b \\ | & & | \\ c & - & a \end{matrix}\right\}$

$\boxed{\#\ \overset{3}{\text{colorings}} \text{ w/ } 2a\text{'s}, 1b, 1c} = \frac{1}{8}\left(\binom{4}{2,1,1}+0+0+2\cdot 2\right) = \frac{1}{8}\left(\frac{4!}{2!1!1!}+4\right) = \frac{1}{8}(12+4) = \overset{\shortparallel}{2}$ //

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<u>Proof of Pólya's thm</u> The main result behind it is on

<u>Burnside's Lemma</u> For a group G of permutations of a finite set X, $\#G$-orbits $\mathcal{O}$ on $X = \frac{1}{|G|}\sum_{\sigma\in G} \#\{x\in X : \sigma(x)=x\}$.

<u>Pf:</u>
$$\sum_{\sigma\in G}\#\{x\in X:\sigma(x)=x\} = \#\{(\sigma,x)\in G\times X : \sigma(x)=x\}$$
$$= \sum_{x\in X} \underbrace{\#\{\sigma\in G : \sigma(x)=x\}}_{G_x := G\text{-stabilizer of } x}$$
$$= \sum_{\substack{G\text{-orbits}\\ \mathcal{O}\text{ on } X}} \underbrace{\sum_{x\in\mathcal{O}} |G_x|}_{|\mathcal{O}||G_x| = |G|}$$
orbit-stabilizer lemma i.e. $|\mathcal{O}| = |G|/|G_x| = [G:G_x]$
$$= |\mathcal{O}|\cdot|G|.$$
▨

When G permutes X, it also permutes k-colorings of X,

and $\sigma\in G$ fixes a k-coloring $\iff$ the k-coloring is constant within cycles of σ

e.g. $X =$ 

$\left|\begin{matrix} a & & b & c & \\ \| \; a & & \| & \| & a \\ a & & b & c & \end{matrix}\right|\begin{matrix} b & & c & c & \\ & b & \| & \| & a \\ b & & c & c & \end{matrix}\Big| \cdots = (a^3+b^3+c^3)\cdot(a^2+b^2+c^2)^2\cdot(a+b+c)$

↑ 3-coloring fixed by σ

Hence $\displaystyle\sum_{\substack{k\text{-colorings}\\ \text{fixed by }\sigma}} a_1^{\#\text{color 1 used}} a_2^{\#\text{color 2 used}}\cdots = \prod_{\substack{\text{cycles}\\ C \text{ of } \sigma}} (a_1^{|C|}+a_2^{|C|}+\cdots+a_k^{|C|}) = \left[t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_j = a_1^j+\cdots+a_k^j}$

Hence <u>pattern inventory</u> $\displaystyle = \sum_{\substack{G\text{-orbits}\\ \mathcal{O}\text{ of colorings}}} \underline{a}^{\text{colors in }\mathcal{O}} = \sum_{\underline{c}=(c_1,\ldots,c_k)} \underline{a}^{\underline{c}}\cdot\#\left\{\begin{matrix}G\text{-orbits } \mathcal{O}\\ \text{using } \underline{c}\end{matrix}\right\} \overset{\text{Burnside}}{=} \sum_{\underline{c}} \underline{a}^{\underline{c}} \frac{1}{|G|}\underbrace{\sum_{\sigma\in G}\#\left\{\begin{matrix}k\text{-colorings}\\ \text{using } \underline{c} \text{ fixed}\\ \text{by } \sigma\end{matrix}\right\}}_{[\underline{a}^{\underline{c}}]\left[t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_j = a_1^j+\cdots+a_k^j}}$

$$= \left[\frac{1}{|G|}\sum_{\sigma\in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_j = a_1^j+a_2^j+\cdots+a_k^j}$$

▨

Some theory of ordinary generating functions (Ardila §2.2)

Roughly speaking, if $\mathcal{A}$ is some class of combinatorial structures, w/ $a_n = \#$ (weighted?) $\mathcal{A}$-structures of wt/size $n \in R$ (a com. ring w/ 1), then we can form the ordinary generating function (o.g.f.) $A(x) = \sum_{n \geq 0} a_n x^n \in R[[x]]$.

Prop. • If $\mathcal{C}$ structures of size n are a choice of either an $\mathcal{A}$- or $\mathcal{B}$- structure of size n (i.e., $c_n = a_n + b_n$) ("$\mathcal{C} = \mathcal{A} + \mathcal{B}$") then $C(x) = A(x) + B(x)$.

- If $\mathcal{C}$-structures of size n are a choice of
  - an $\mathcal{A}$-structure of size i
  - a $\mathcal{B}$- structure of size j

  for some $i + j = n$ ("$\mathcal{C} = \mathcal{A} \times \mathcal{B}$")

  (i.e. $c_n = \sum_{\substack{i+j=n \\ i,j \geq 0}} a_i b_j$)

  then $C(x) = A(x) \cdot B(x)$.

- If $\mathcal{C}$-structures of size n are a choice of $\mathcal{B}$-structures of sizes $i_1, i_2, \ldots, i_k$ for some $i_1 + i_2 + \cdots + i_k = n$, $i_j \geq 0$

  (i.e., $c_n = \sum_{\substack{(i_1, i_2, \ldots, i_k), \\ \sum i_j = n, \\ i_j \geq 0}} b_{i_1} b_{i_2} \cdots b_{i_k}$) ("$\mathcal{C} = \mathrm{Seq}(\mathcal{B})$" $\downarrow$ "sequence")

  then $C(x) = \dfrac{1}{1 - B(x)}$.

Pf is straightforward. Let's see several examples of how to apply this proposition ...

EXAMPLES (see also Ardila §2.2.2)

(Partitions w/ bounded part size)

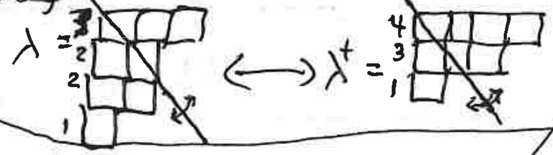
① Let $p_k(n) := \#\{\text{partitions } \lambda \vdash n \text{ w/ } \lambda_1 \le k\}$ (so $\lambda_i \le k \; \forall i$)

$(\lambda_1 \ge \lambda_2 \ge \cdots)$ (in bijection via conjugation of partitions)

$= \#\{\text{partitions } \lambda \vdash n \text{ w/ } \ell(\lambda) \le k\}$

$\lambda \Leftrightarrow \lambda^t$ [flip Young diagram along main diagonal]

example of conjugation:



Then $P_k(q) = \sum_{n \ge 0} p_k(n) q^n \left(= \sum_{\lambda : \lambda_1 \le k} q^{|\lambda|} = \sum_{\lambda : \ell(\lambda) \le k} q^{|\lambda|} \right)$

$$= \underbrace{\frac{1}{1-q}}_{\text{o.g.f. for } \lambda \text{ w/ only parts of size 1}} \cdot \underbrace{\frac{1}{1-q^2}}_{\text{o.g.f. for } \lambda \text{ w/ only parts of size 2}} \cdot \cdots \cdot \underbrace{\frac{1}{1-q^k}}_{\ldots \text{ w/ only parts of size } k} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

we've seen the $k \to \infty$ limit of this earlier

i.e. $\mathcal{C} = \{\lambda \text{ w/ } \lambda_1 \le k\} = \text{Seq}(\text{Ones}) \times \text{Seq}(\text{Twos}) \times \cdots \times \text{Seq}(k\text{'s})$

Similarly, $\sum_{\lambda : \lambda_1 \le k} q^{|\lambda|} t^{\ell(\lambda)} = \frac{1}{(1-tq)(1-tq^2)\cdots(1-tq^k)} \left(= \sum_{\lambda : \ell(\lambda) \le k} q^{|\lambda|} t^{\lambda_1} \right)$

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② (Compositions)

DEF'N A composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n, denoted $\alpha \models n$, is a sequence of positive integers $\alpha_i \in \{1, 2, \cdots\}$ w/ $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$.
(Unlike partitions, no requirement that α_i weakly decrease).

As w/ partitions, α_i are called the parts of α.

Prop # compositions $\alpha \models n$ of n into k parts $= \binom{n-1}{k-1}$.

$(\alpha_1, \alpha_2, \ldots, \alpha_k)$

Pf: Say $\alpha = (\alpha_1, \dots, \alpha_k)$ is a weak composition of n if α_i are nonnegative integers $\alpha_i \in \{0,1,2,\dots\}$ w/ $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$.

weak compositions of n into k parts $= \left(\!\!\binom{k}{n}\!\!\right)$ (recall = # size n multisets of $[k]$)

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \longleftrightarrow \{\underbrace{1,1,\dots,1}_{\alpha_1}, \underbrace{2,2,\dots,2}_{\alpha_2}, \dots, \underbrace{k,k,\dots k}_{\alpha_k}\}$$

also,

weak compositions of n into k parts = # (usual) compositions of $(n+k)$ into k parts

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \longleftrightarrow (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_k)$$

Hence # comp. of n into k parts = # weak comp. of $(n-k)$ into k parts $= \left(\!\!\binom{k}{n-k}\!\!\right) = \binom{k+n-k-1}{n-k} = \binom{n-1}{n-k} = \binom{n-1}{k-1}$ ☒

Thus, total # compositions of n $= \sum_{k=1}^{n}$ # comp $\alpha \vDash n$ into k parts $= \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$. (for $n=0$, get 1 instead)

Let's see a different way to get this using o.g.f.'s:

Let $a_n :=$ # {comp. $\alpha = (\alpha_1, \dots, \alpha_k) \vDash n$ of any length k} (convention $a_0 := 1$)

$$\text{Then } A(x) = \sum_{n \geq 0} a_n x^n = \frac{1}{1 - \underbrace{(x + x^2 + x^3 + \cdots)}} = \frac{1}{1 - \frac{x}{1-x}}$$

"$A = Seq$(one-part compositions)"

o.g.f. for compositions of n w/ one part since ∃ unique comp. of n into one part

$$= \frac{1-x}{1-2x}$$

$$= 1 + \frac{x}{1-2x}$$

$$= 1 + \sum_{n \geq 1} 2^{n-1} x^n. \checkmark$$

(compositions into odd parts)

③ What about $a_n = \{$compositions $\alpha \models n$ into odd parts$\}$?

n	$\alpha \models n$ w/ odd parts	$\#\{\alpha\}$
0	()	1
1	1	1
2	1+1	1
3	3, 1+1	2
4	3+1, 1+3, 1+1+1+1	3
5	5, 3+1+1, 1+3+1 1+1+3, 1+1+1+1+1	5
6	~	8

Guess $\alpha_n = \begin{cases} 1 & n=0 \\ F_{n-1} & n \geq 1 \end{cases}$ Recall Fibonacci #

and indeed

$$A(x) = \sum_{n\geq 0} a_n x^n = \frac{1}{1-(x^1+x^3+x^5+\dots)}$$

"$A = Seq($one odd part compositions$)$" $= \dfrac{1}{1-\frac{x}{1-x^2}} = \dfrac{1-x^2}{1-x^2-x}$

saw this earlier

$$= 1 + \frac{x}{1-x-x^2} = 1 + \sum_{n\geq 1} F_{n-1} x^n$$

④ Stirling #'s of the 2nd kind

DEF'N A set partition of $[n]$ is a set $\pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ of subsets $\pi_i \subseteq [n]$ s.t.

- (nonempty) $\pi_i \neq \emptyset \ \forall i$
- (disjoint) $\pi_i \cap \pi_j = \emptyset \ \forall i \neq j$
- (covering) $\bigcup \pi_i = [n]$.

The π_i are called the blocks of the set partition π.

DEF'N $S(n,k)$:= # set partitions of $[n]$ into exactly k blocks
for $1 \leq k \leq n$ "Stirling #'s of 2nd kind."

$S(n,k)$ = rank sizes for the poset $(\Pi_n, \leq)$

$\Pi_n = \{$all set partitions of $[n]\}$; $\leq$ = refinement, i.e. $\pi^1 \leq \pi^2$ iff π^1 refines π^2

RIEF ASIDE: Π_n is the lattice of intersections
the Braid hyperplane arrangement: $\{\{x_i = x_j\} : 1 \leq i < j \leq n\}$

e.g. of partition lattice Π_n

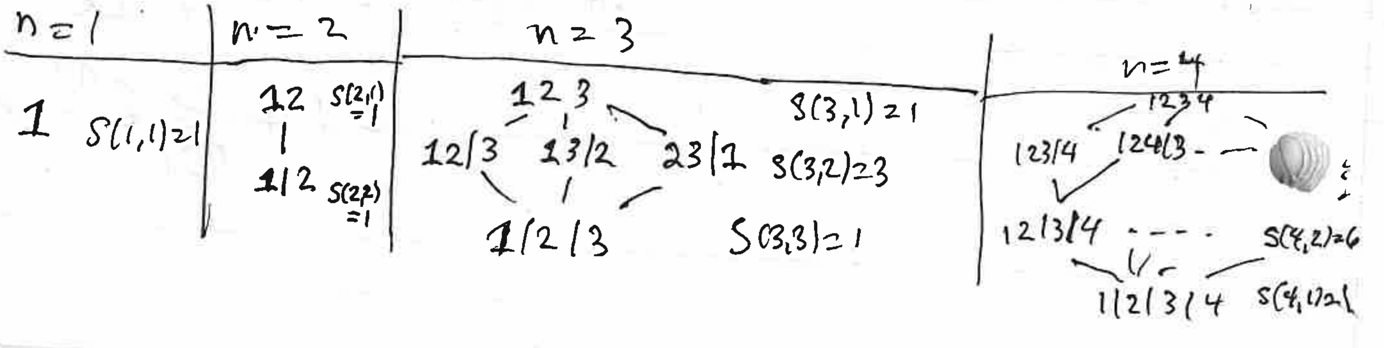


Table of $S(n,k)$:

$n \backslash k$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	1	3	1	0
4	1	7	6	1
				...

(Pascal-like) Recurrence for $S(n,k)$:

$$S(n,k) = \underbrace{S(n-1,k-1)}_{\substack{n \text{ is a singleton} \\ \text{block}}} + \underbrace{k\, S(n-1,k)}_{\substack{n \text{ goes into} \\ \text{one of the} \\ k \text{ other blocks}}} \text{ for } k>1$$

w/ initial conditions $S(n,1)=1\ \forall n$ and $S(0,0)=1$

9/30 Let's Study the o.g.f. $F_k(x) = \sum_{n\geq 0} S(n,k)x^n$ in 2 ways.

(a) Solve recurrence: for $k\geq 2$

$$\sum_{n\geq 0} S(n,k)\,x^n = \sum_{n\geq 0} S(n-1,k-1)x^n + \sum_{n\geq 0} k\,S(n-1,k)\,x^n$$

$$F_k(x) = x\,F_{k-1}(x) + kx\,F_k(x)$$

$$(1-kx)F_k(x) = x\,F_{k-1}(x)$$

$$\boxed{F_k(x) = \frac{x}{1-kx}F_{k-1}(x)}$$

$\left(\text{and for } k=1,\ F_1(x) = \sum_{n\geq 0} S(n,1)x^n = x+x^2+x^3+\cdots = \frac{x}{1-x}\right)$

$$\Rightarrow F_k(x) = \frac{x}{1-kx}\cdot\frac{x}{1-(k-1)x}\cdots\frac{x}{1-2x}\cdot\frac{x}{1-x} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$ //

(b) Let $A_m :=$ the structure of strings of letters from $[m]$ that start w/ an m, whose <u>size</u> is their <u>length</u> $\left(\text{e.g. } m=3 \quad \underbrace{\underline{3}\,1312}_{\text{size } 5} \text{ or } \underbrace{\underline{3}\,311}_{\text{size } 4}\right)$

<u>Prop</u>: $\left\{\begin{matrix}\text{set partitions} \\ \pi \text{ of } [n] \text{ w/ } k \text{ blocks}\end{matrix}\right\} \xleftrightarrow{\text{bijection}} \left\{\begin{matrix}\text{total size} = n \\ \text{structures in } A_1 \times A_2 \times \cdots \times A_k\end{matrix}\right\}$

$\pi \longmapsto$ the <u>restricted growth function</u> $[n]\xrightarrow{f}[k]$ associated to π

e.g., $n=16$, $k=4$

$$\overset{①}{\underline{1},2,4,5,8,12} \,\Big|\, \overset{②}{\underline{3},6,9,10,14} \,\Big/\, \overset{③}{\underline{7},11,16} \,\Big|\, \overset{④}{\underline{13},15} \mapsto$$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(i)$	①	①	②	①	①	②	③	①	②	②	③	①	④	②	④	③
	$\in A_1$		$\in A_2$				$\in A_3$						$\in A_4$			

number the blocks of π ①, ②, ..., ⓚ according to <u>increasing smallest elements</u>

$f(i) :=$ block # of i

<u>Pf</u>: Exercise. ∎

<u>Cor</u>

$$F_k(x) = \overbrace{\underset{\substack{\shortparallel \\ x+x^2+x^3+\cdots}}{\frac{x}{1-x}}}^{\text{o.g.f. for } A_1}\cdot\overbrace{\underset{\substack{\shortparallel \\ x+2x^2+4x^3+\cdots}}{\frac{x}{1-2x}}}^{\text{o.g.f. for } A_2}\cdots\overbrace{\underset{\substack{\shortparallel \\ x+kx^2+k^2x^3+\cdots}}{\frac{x}{1-kx}}}^{\text{o.g.f. for } A_k} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$ //

Digression on the two kinds of Stirling #'s

How are $S(n,k)$ and $c(n,k)$ related?

$S(n,k)$: stirling #'s of 2nd kind

$c(n,k) := \#\{\sigma \in \mathfrak{S}_n : \#\text{cycles}(\sigma) = k\}$: (signless) stirling #'s of 1st kind

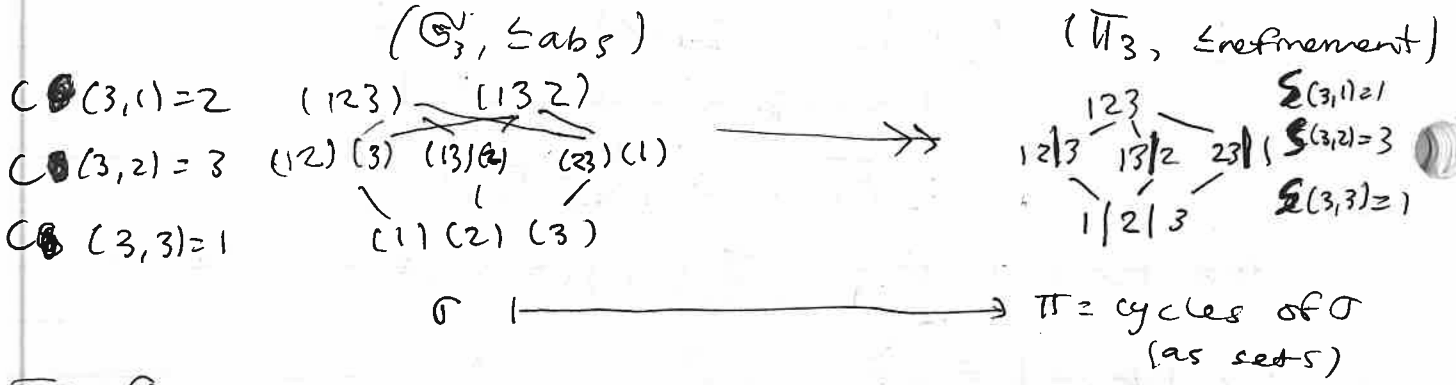
$c(n,k)$

$n \backslash k$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	2	3	1	0
4	6	11	6	1

ⓐ The $c(n,k)$ satisfy a similar recurrence:

$$c(n,k) = \underbrace{c(n-1,k-1)}_{\substack{n \text{ goes in} \\ 1\text{-cycle}}} + \underbrace{(n-1)\,c(n-1,k)}_{\substack{n \text{ maps to} \\ \text{some } i \in [n-1]}}$$

ⓑ They are rank #'s for posets w/ an order + rank-preserving surjection between them:



ⓐ+ⓑ are more superficial. The real relation btween $S(n,k)$ & $c(n,k)$ is...

ⓒ **Prop** (i) $x^n = \sum_{k=1}^{n} S(n,k)\,(x)_k$ where $(x)_k := x(x-1)(x-2)\cdots(x-(k-1))$

while (ii) $(x)_n = \sum_{k=1}^{n} \underset{\substack{\shortparallel \\ (-1)^{n-k} c(n,k)}}{s(n,k)}\, x^k$ (= (signed) stirling #'s of 1st kind)

and hence (iii) the infinite (uni lower triangular) matrices

$$(S(n,k))_{\substack{k=0,1,2,\ldots \\ \cancel{k=1,2,\ldots} \\ n=\cancel{1}0,1,2,\ldots}} \text{ and } (s(n,k))_{\substack{\cancel{k=1,2,\ldots} \\ \cancel{n=1,2,\ldots}}} \; {}_{\substack{k=0,1,2,\ldots \\ n=0,1,2,\ldots}}$$

give the inverse change-of-basis matrices between the ordered bases $\{x^n\}_{n=0,1,2,\ldots}$ and $\{(x)_n\}_{n=0,1,2,\ldots}$ of $\mathbb{C}[x]$.

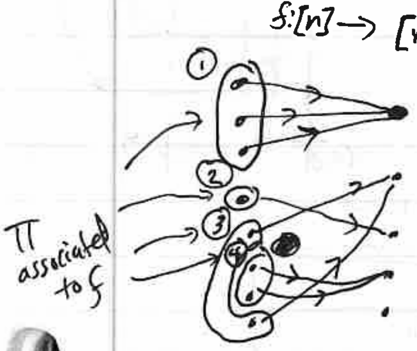
in particular, (iv) $\sum_{k=1}^{n} S(n,k)\,s(k,m) = \underset{\text{Kroenecker delta}}{\delta_{n,m}} = \sum_{k=1}^{n} s(n,k)\,S(k,m)$.

Pf of prop:

For (i), note that both sides lie in $\mathbb{C}[x]$ (of degree n), so it is enough to prove equality holds for $x=1,2,3,\ldots$ (since a polynomial $\underset{\text{LHS-RHS}}{f(x)} \in \mathbb{C}[x]$ that vanishes at $x=1,2,3,\ldots$ must be $f\equiv 0$)] - useful proof technique.

For $x=1,2,3,\ldots$

$$x^n = \#\left\{\begin{matrix}\text{functions}\\ [n]\xrightarrow{f}[x]\end{matrix}\right\} = \sum_{\substack{\text{set partitions}\\ \pi \text{ of } [n]}} \#\left\{\begin{matrix} f:[n]\to[x] \text{ for which}\\ \text{set of (non-empty) fibers } \{\underset{\neq\emptyset}{f^{-1}(i)}\}_{i\in[x]} = \pi\end{matrix}\right\}$$



choices of which $i\in[x]$ are images of the (non-empty) fibers determined by π

$$= \sum_{k=1}^{n} S(n,k)\, \underbrace{x(x-1)(x-2)\cdots(x-(k-1))}_{(x)_k}$$

For (ii), recall: $x(x+1)(x+2)\cdots(x+(n-1)) = \sum_{k=1}^{n} c(n,k)\,x^k$

$\Downarrow$ $x\mapsto -x$, and multiply by $(-1)^n$

$$x(x-1)(x-2)\cdots(x-(n-1)) = \sum_{k=1}^{n} \underbrace{(-1)^{n-k}c(n,k)}_{s(n,k)}\, x^k \quad ✓$$

Then (iii) + (iv) follow ... ∎

10/2 Back to o.g.f examples ...

⑤ Let $a_n := \#\{\sigma\in\mathfrak{S}_n : \sigma$ is irreducible / indecomposable $\}$ for $n\geq 1$

i.e. it can't be factored as $\sigma = \underset{\mathfrak{S}_{\{1,2,\ldots,k\}}}{\sigma_1} \; \underset{\mathfrak{S}_{\{k+1,k+2,\ldots,n\}}}{\sigma_2}$ for $1\leq k<n$

e.g. $\sigma=(135)(2)(4)\in\mathfrak{S}_5$ is irreducible

but $\sigma = \underbrace{(13)(2)}_{\in\mathfrak{S}_{\{1,2,3\}}}\underbrace{(45)}_{\mathfrak{S}_{\{4,5\}}}$ is not!

n	σ irreducible in $\mathfrak{S}_n$	a_n
1	(1)	1
2	(12) ~~(1)(2)~~	1
3	(123), (132), (13)(2)	3
4	...	13

Q: How to compute a_n?

Note: Permutations = Seq(irreducible permutations)

So if we let $A(x) = \sum_{n\geq1} a_n x^n$

and $B(x) = \sum_{n\geq0} \underbrace{n!}_{\#\mathfrak{S}_n} x^n$ ($\in \mathbb{C}[[x]]$ but no radius of convergence!)

then $B(x) = \dfrac{1}{1-A(x)}$ and so

use some computer algebra software...

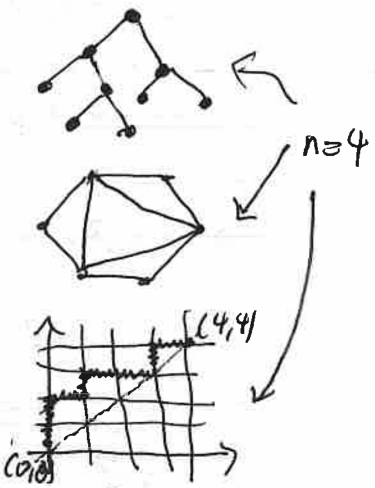
$$A(x) = 1-\frac{1}{B(x)} = 1-\frac{1}{\sum_{n\geq0} n!\,x^n} = x+x^2+3x^3+13x^4+71x^5+\cdots$$

⑥ The Catalan family (see Stanley's other book on this subject!)

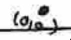
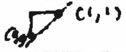
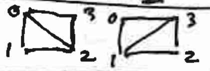
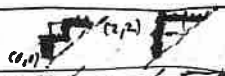
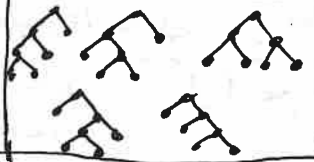
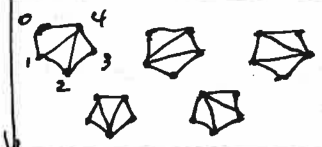
$C_n :=$ Catalan number $= \#\left\{\begin{array}{l}\text{plane binary trees} \\ \text{w/ } n+1 \text{ leaves (or} \\ n \text{ internal vertices,} \\ \text{each w/ a left + right child)}\end{array}\right\}$

$= \#\{\text{triangulations of } (n+2)\text{-gon}\}$

$= \#\left\{\begin{array}{l}\text{lattice paths taking } N, E \\ \text{steps } (0,0) \to (n,n), \text{ staying} \\ \text{(weakly) above diagonal } y = x\end{array}\right\}$



Theorem $C_n = \frac{1}{n+1}\binom{2n}{n}\left(= \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{2n+1}\binom{2n+1}{n}\right)$

n	C_n	plane binary trees	triangulations	lattice paths
0	$1=\frac{1}{1}\binom{0}{0}$			
1	$1=\frac{1}{2}\binom{2}{1}$			
2	$2=\frac{1}{3}\binom{4}{2}$			
3	$5=\frac{1}{4}\binom{6}{3}$			
4	$14=\frac{1}{5}\binom{8}{4}$	. . .	. . .	(0,0) . . .

Pf of thm: $C(x) := \sum_{n\geq 0} C_n x^n = 1 + \sum_{n\geq 1} C_n x^n$

$= 1 + \underbrace{C(x)\cdot x\cdot C(x)}$

fundamental recurrence

i.e., $C_n = \sum_{i+j=n-1} C_i C_j$ for $n \geq 1$

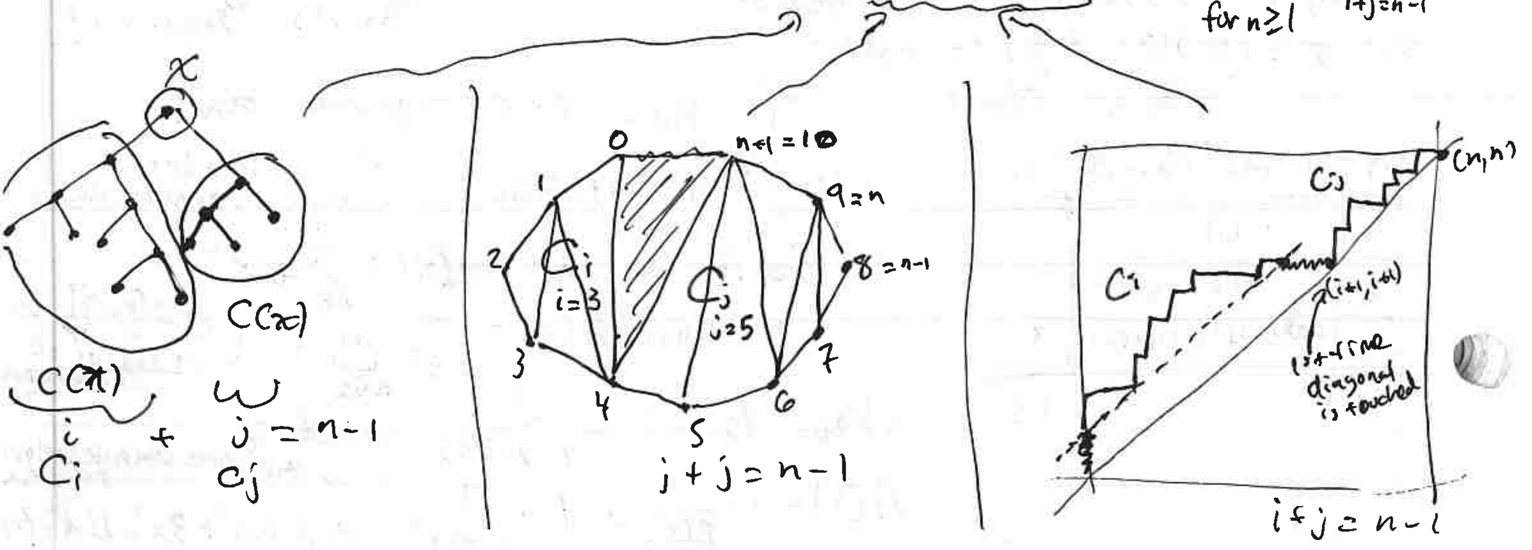


figure out $\pm\sqrt{1-4x}$ by expanding $\sqrt{1-4x}$

Consequently, $C(x) = 1 + x\,C(x)^2 \Rightarrow 0 = x\,C(x)^2 - C(x) + 1 \Rightarrow C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n = \sum_{n\geq 0} \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})\cdots(\frac{-(2n-3)}{2})}{n!}(-1)^n 4^n x^n$$

$$= 1 - 2\sum_{n\geq 1} 2^{n-1}\frac{(1)(3)\cdots(2n-3)}{n!}x^n$$

$$= 1 - 2x\sum_{n\geq 1}\frac{(2)(4)\cdots(2n-2)}{(n-1)!}\cdot\frac{(1)(3)\cdots(2n-3)}{n!}x^{n-1}$$

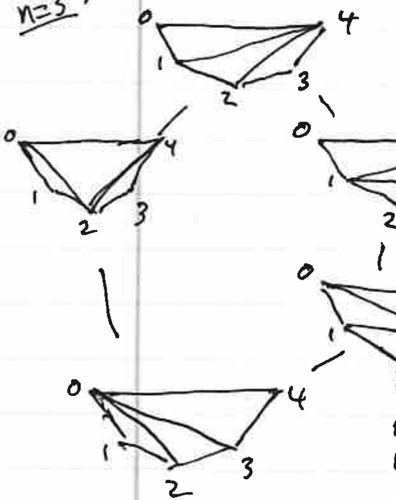
$$= 1 - 2x\sum_{n\geq 1}\frac{1}{n}\binom{2n-2}{n-1}$$

$$\Rightarrow C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 1}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1} = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n.$$

Aside: There are (at least) 3 different interesting posets on Catalan objects:

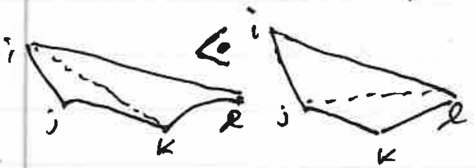
Tamari lattice on triangulations of $(n+2)$-gon

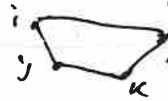
$n=3$:



not ranked!

Cover relations:



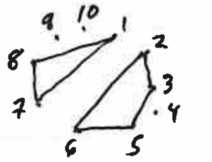
Switch ik diagonal to jl diagonal in  quadrilateral

The interval $[\emptyset,$  $]$ in Young's lattice $\mathbb{Y}$ of partitions:

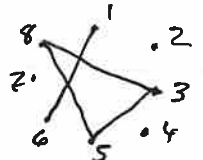
$n=3$:

ranked!

$NC(n) :=$ noncrossing set partitions of (w.r.t. $\leq$ refinement)

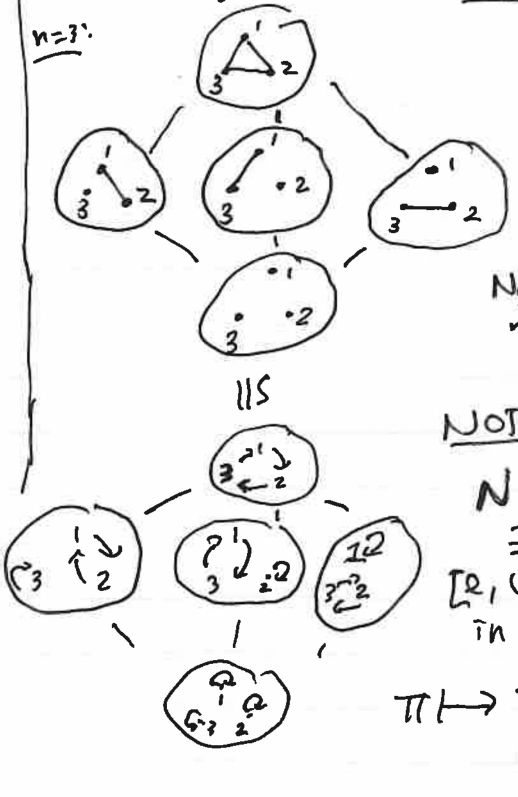


178/2356/4/9/10 is noncrossing!



16/2/358/4/7 is crossing!

$n=3$:



ranked

rank # are cal

Narayan numbers

NOTE!

$NC(n) \cong [e, (123\cdots n)]$ in absolute order on

$\pi \mapsto$ orient cycles of counter-clockwi