Exponential generating functions (Ardila § 2.3)

$\mathcal{A}$ = structure one can place on labelled objects like $[n]$

a_n = # of such structures one can place on $[n]$

$\rightsquigarrow A(x) := \sum_{n\geq 0} a_n \frac{x^n}{n!}$ =: exponential gen. fn for $\mathcal{A}$

Prop:
- If $\mathcal{C}$ structures are a choice of $\mathcal{A}$- or $\mathcal{B}$-structures, ("$\mathcal{C}=\mathcal{A}+\mathcal{B}$")
  then $C(x) = A(x) + B(x)$
- If $\mathcal{C}$-structures on $[n]$ are a choice of a partition $[n] = S_1 \sqcup S_2$, with an $\mathcal{A}$-structure on S_1, $\mathcal{B}$-structure on S_2 ("$\mathcal{C}=\mathcal{A}*\mathcal{B}$")
  so that $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, then $C(x) = A(x)B(x)$.
- If $\mathcal{C}$-structures are a choice of (unordered) set partition π of $[n]$, and then an $\mathcal{A}$-structure on each block of π ("$\mathcal{C}=Set(\mathcal{A})$")
  then $\boxed{C(x) = e^{A(x)}}$

anical pf.: ecd $a_0=0$ for this make sense

The exponential formula

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Pf:
- $\mathcal{C}=\mathcal{A}+\mathcal{B}$ is obvious.
- For $\mathcal{C}=\mathcal{A}*\mathcal{B}$, note $c_n = \sum_{i=0}^{n}\binom{n}{i} a_i b_{n-i} \Leftrightarrow \frac{c_n}{n!} = \sum_{i+j=n} \frac{a_i}{i!}\frac{b_j}{j!}$

  $\Leftrightarrow C(x) = A(x)B(x)$. ✓
- For $\mathcal{C}=Set(\mathcal{A})$, note $\mathcal{C} = \bigsqcup_{k=1}^{\infty} \mathcal{A}^{(k)}$, where $\mathcal{A}^{(k)}$ = {pick a set partition π into exactly k (unordered) blocks and put an $\mathcal{A}$-structure on each block}

So $C(x) = \sum_{k=1}^{\infty} A^{(k)}(x)$.

But $k!\,A^{(k)}(x) = A(x)^k$ = e.g.f. for $\underbrace{\mathcal{A}*\mathcal{A}*\cdots*\mathcal{A}}_{k \text{ times}}$ = {pick a set partition $\pi = B_1 \sqcup \cdots \sqcup B_k$ into k ordered blocks, and put an $\mathcal{A}$-structure on each block}.

Hence $A^{(k)}(x) = \frac{A(x)^k}{k!}$

and so $C(x) = \sum_{k=1}^{\infty} \frac{A(x)^k}{k!} = e^{A(x)}$. ▨

NOTE: $a_0 = 0 \Rightarrow$ all these $B_i \neq \emptyset$.

hat-check probability

EXAMPLES:

① Recall $d_n = \#\{\text{derangements in } \mathfrak{S}_n\}$, $D(x) := \sum_{n\geq 0} \frac{d_n}{n!} x^n$

$$\left\{\begin{matrix}\text{all}\\ \text{permutations}\end{matrix}\right\} = \left\{\begin{matrix}\text{fixed point only perms,}\\ \text{i.e., identity perms}\end{matrix}\right\} * \left\{\begin{matrix}\text{derangements}\\ \text{(fixed-pt-free perms}\end{matrix}\right\}$$

So $\sum_{n\geq 0} \cancel{n!} \frac{x^n}{\cancel{n!}} = \left(\sum_{n\geq 0} 1 \cdot \frac{x^n}{n!}\right) \cdot D(x)$

$$\frac{1}{1-x} = e^x \cdot D(x), \text{ i.e., } D(x) = \frac{e^{-x}}{1-x}, \text{ as we saw}$$

② $\left\{\begin{matrix}\text{involutions}\\ \sigma^2 = 1\end{matrix}\right\} = \text{Set}\left(\left\{\begin{matrix}\text{involutions w/}\\ \text{exactly one cycle}\end{matrix}\right\}\right)$

Hence $\sum_{n\geq 0} \#\left\{\begin{matrix}\sigma \in \mathfrak{S}_n:\\ \sigma^2 = 1\end{matrix}\right\} \frac{x^n}{n!} = e^{\sum_{n\geq 0} \#\{\sigma \in \mathfrak{S}_n : \sigma^2 = 1, \#\text{cycles}(\sigma) = 1\} \frac{x^n}{n!}}$

$$= e^{0 \cdot \frac{x^0}{0!} + 1 \cdot \frac{x^1}{1!} + \frac{1 \cdot x^2}{2!} + \frac{0 \cdot x^3}{3!} + \frac{0 \cdot x^4}{4!} + \cdots}$$

$$= e^{x + \frac{x^2}{2}}, \text{ as we saw before}$$

③ More generally Touchard's THM follows from exp. formula:

$\{\text{permutations}\} = \text{Set}(\{\text{permutations w/ exactly one cycle}\})$

and if we weight σ by $t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots$, wt is multiplicative with respect to this decomposition.

So $\sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots\right) = e^{\sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{\substack{\sigma \in \mathfrak{S}_n: \\ \sigma \text{ has exactly} \\ \text{one cycle}}} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots\right)}$

$$= e^{\sum_{n\geq 1} \frac{x^n}{n!} \cdot t_n (n-1)!}$$

there are $(n-1)!$ n-cycles in $\mathfrak{S}_n$ $(1\ \underbrace{a_1\ a_2 \cdots a_{n-1}}_{\text{arbitrary seq.}})$

$$= e^{\sum_{n\geq 1} t_n \frac{x^n}{n}}$$

$$= e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots}, \text{ as we saw.}$$

In addition to permutations, e.g.f.'s useful for set partitions and graphs/trees ...

④ **Bell numbers** $B_n := \#\{\text{set partitions of } [n]\}$

+ **Bell polynomials** $B_n(y) := \sum_{\substack{\text{set partitions}\\ \pi \text{ of } [n]}} y^{\#\text{blocks}(\pi)} = \sum_{k \geq 1}^{n} S(n,k) y^k$

Since $\{\text{set partitions}\} = \text{Set}(\{\text{single (non-empty) block partitions}\})$,

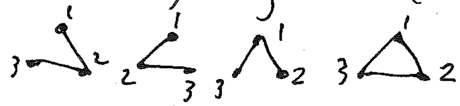
$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{1\cdot\frac{x}{1!} + 1\cdot\frac{x^2}{2!} + 1\cdot\frac{x^3}{3!} + \cdots} = e^{(e^x - 1)},$$

and

$$\sum_{n\geq 0} B_n(y) \frac{x^n}{n!} = e^{y\cdot\frac{x}{1!} + y\cdot\frac{x^2}{2!} + y\frac{x^3}{3!} + \cdots} = e^{y(e^x-1)}$$

Cor (extract coeff. of $[y^k]$) $\Rightarrow \sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$

10/9 ⑤ Let's count connected, simple graphs $G = (\overset{[n]}{\overset{\shortparallel}{V}}, E \subseteq \binom{[n]}{2})$, weighted by $y^{|E|}$ (number of edges).

e.g. $n=3$,  $\Rightarrow 3y^2 + y^3$

Can we understand $\text{Conn}(x,y) := \sum_{n\geq 1} \frac{x^n}{n!} \sum_{\substack{\text{connected simple}\\ \text{graphs } G \text{ on } [n]}} y^{|E(G)|}$?

NOTE: $\{\text{all simple graphs}\} = \text{Set}(\{\text{connected simple graphs}\})$

So $\text{All}(x,y) = e^{\text{Conn}(x,y)}$

$$\Rightarrow \text{Conn}(x,y) = \log(\text{All}(x,y)) = \log\Big(\sum_{n\geq 0}\frac{x^n}{n!}\sum_{\substack{\text{simple}\\ \text{graphs } G \text{ on } [n]}} y^{|E(G)|}\Big)$$

$$= \log\Big(1 + \sum_{n\geq 1}\frac{x^n}{n!}(1+y)^{\binom{n}{2}}\Big) \quad \leftarrow \text{include each edge or not!}$$

computer $\searrow$

$$= x + \frac{x^2}{2!}\cdot y + \frac{x^3}{3!}(3y^2+y^3) + \frac{x^4}{4!}(16y^3 + 15y^4 + 6y^5 + y^6) + \cdots$$

⑥ Let's try to understand $t_n := \#\{\text{trees on } [n]\}$

n	trees	t_n
1		1
2		1
3		3
4	...	16
5	...	125

If we define $V_n := \#\{\text{vertex-rooted trees on } [n]\}$, $\mathcal{V}$

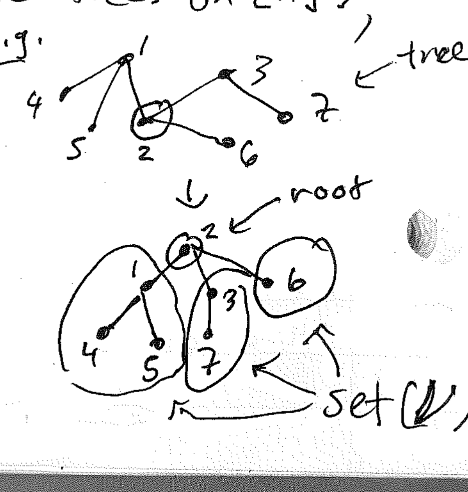
then $V_n = n \cdot t_n$

and $\mathcal{V} = \{\text{root}\} * \text{Set}(\mathcal{V})$

So that $\boxed{V(x) = x e^{V(x)}}$

$V(x) = \sum_{n\geq 0} V_n \frac{x^n}{n!}$

Is this useful? Yes! ...



Can rephrase as $V(x)\, e^{-V(x)} = x$, or in other words,
$V(x)$ is the compositional inverse to $A(x) = xe^{-x}$ in $\mathbb{C}[[x]]$.

((easy) Prop: If $A(x) = a_1 x + a_2 x^2 + \cdots \in R[[x]]$ has no constant term ($a_0 = 0$),
so that $B(A(x))$ is well-defined, then A has a compositional inverse
$B = A^{\langle -1 \rangle}$, satisfying $B(A(x)) = x$ (and $A(B(x)) = x$ by associativity of $A \circ B$)
$\iff$ $a_1 \in R^\times$ is a unit.)

Why does knowing $V(x) = A^{\langle -1 \rangle}(x)$ for $A(x) = xe^{x}$ help?

Lagrange inversion thm:

If $B(x) = A^{\langle -1 \rangle}(x)$, that is, $B(A(x)) = x$ for some $A(x), B(x) \in x\mathbb{C}[[x]]$,
then $[x^n]\, B(x) = \frac{1}{n} [x^{-1}] \left(\frac{1}{A(x)^n} \right) \left(= \frac{1}{n} [x^{n-1}] \left(\frac{x}{A(x)} \right)^n \right)$.

Before we prove this thm, let's see some examples...

EXAMPLE:

(a) $V(x) = \sum_{n \geq 0} v_n \frac{x^n}{n!}$ where v_n = # vertex-rooted trees on $[n]$ $= n \cdot t_n$

has $V(x) = A^{\langle -1 \rangle}(x)$ for $A(x) = xe^{-x}$.

So $\frac{v_n}{n!} = [x^n]\, V(x) = \frac{1}{n} [x^{n-1}] \left(\frac{x}{xe^{-x}} \right)^n = \frac{1}{n} [x^{n-1}]\, e^{nx} = \frac{1}{n} \frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$

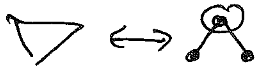
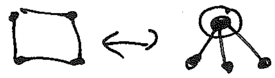
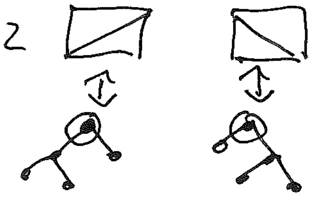
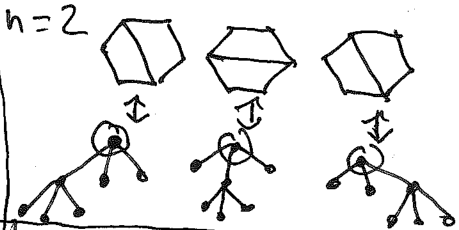
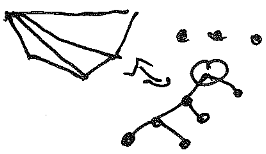
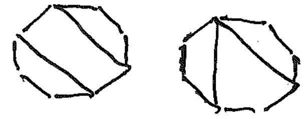
$\Rightarrow v_n = n^{n-1}$, and hence $\boxed{t_n = \frac{v_n}{n} = n^{n-2}}$ ← Cayley's formula

(b) Generalizing Catalan #'s C_n, let's define the

Fuss-Catalan # $C_n^{(k)} :=$ #{k-ary rooted plane trees with n internal vertices}
↳ each having k children ordered left-to-right.

$= \#\{(k+1)\text{-angulations of } ((k-1)n+2)\text{-gon}\}$

$k=2$ ⟶ C_n, usual Catalan #

e.g. $k=2$	$C_n^{(2)}=C_n$	$k=3$	$C_n^{(3)}$
$n=1$ 	$1=\frac12\binom21$	$n=1$ 	$1=\frac13\binom31$
$n=2$ 	$2=\frac13\binom42$	$n=2$ 	$3=\frac15\binom62$
$n=3$ 	$5=\frac14\binom63$	$n=3$  ...	$12=\frac17\binom93$

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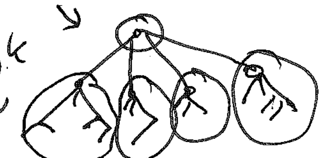
Thm $C_n^{(k)} = \frac{1}{(k-1)n+1}\binom{kn}{n}$ $\left(\overset{k=2}{\rightsquigarrow} \frac{1}{n+1}\binom{2n}{n} = C_n \right)$

Pf: $C(x) := \sum_{n\geq 0} C_n^{(k)} x^n$ satisfies $C(x) = 1 + x\,C(x)^k$:

So that $B(x) := C(x) - 1$ $\left(= \sum_{n\geq 1} C_n^{(k)} x^n\right)$ satisfies $B(x) = x(B(x)+1)^k$,



i.e., $\frac{B(x)}{(B(x)+1)^k} = x \iff B(x) = A^{\langle -1\rangle}(x)$ for $A(x) = \frac{x}{(1+x)^k}$.

Hence Lagrange Inversion Thm says:

$$C^{K}_{(n)} = [x^n]\,B(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{x/(1+x)^k}\right)^n = \frac{1}{n}[x^{n-1}]\,(1+x)^{kn}$$

$$= \frac{1}{n}\binom{kn}{n-1} = \frac{(kn)!}{n!\,(kn-(n-1))!} = \frac{(kn)!}{n!\,((k-1)n+1)!} = \frac{1}{((k-1)n+1)}\binom{kn}{n}. \quad \blacksquare$$

Pf of Lagrange Inv. Thm:

Let $B(x) = \sum_{n\geq 1} b_n x^n$ satisfy $x = B(A(x)) = \sum_{m\geq 1} b_m A(x)^m$.

We want to show $b_n = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right)$: d/dx

$$1 = \sum_{m\geq 1} m\, b_m\, A(x)^{m-1} A'(x)$$

divide by $A(x)^n$

$$\frac{1}{A(x)^n} = \sum_{m\geq 1} m b_m A(x)^{m-n-1} A'(x)$$ (working in $x^{-n}\mathbb{C}[[x]]$ or $\bigcup_{n\geq 0} x^{-n}\mathbb{C}[[x]]$ ring of formal Laurent series)

$$\frac{1}{A(x)^n} = \underbrace{n b_n \frac{A'(x)}{A(x)}}_{\text{term } m=n} + \underbrace{\sum_{\substack{m\geq 1,\\ m\neq n}} m b_m \, d/dx\left(\frac{A(x)^{m-n}}{m-n}\right)}_{\text{all other terms}}.$$

take $[x^{-1}]$

$$[x^{-1}]\left(\frac{1}{A(x)^n}\right) = n b_n [x^{-1}]\left(\frac{a_1x^0+2a_2x^1+3a_3x^2+\cdots}{a_1x^1+a_2x^2+a_3x^3+\cdots}\right) + \sum_{\substack{m\geq 1,\\ m\neq n}} (0)$$

$= n b_n$,

i.e., $b_n = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right)$. ∎

Since... **LEMMA** Any Laurent series $f(x) = c_{-n}x^{-n} + c_{-n+1}x^{-n+1} + \cdots$ satisfies $[x^{-1}]f'(x) = 0$

New topic: q-analogs + the q-binomial coefficients

Recall that $\sum_{\text{all } \lambda} q^{|\lambda|} = \sum_{n\geq 0} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$

and $\sum_{\lambda:\, \lambda_1\leq k} q^{|\lambda|} = \sum_{n\geq 0} p_{\leq k}(n) q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$

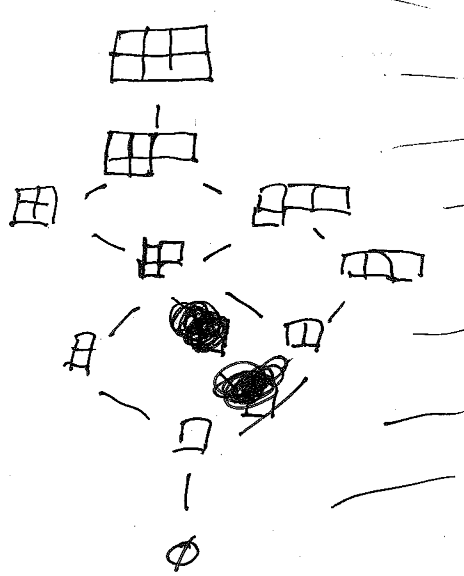
$\lambda \Leftrightarrow \lambda^t$

$$\| \qquad \sum_{\lambda:\, \ell(\lambda)\leq k} q^{|\lambda|}$$

Q: What about $\begin{bmatrix} j+k \\ k \end{bmatrix}_q := \sum_{\substack{\lambda:\\ \lambda_1\leq j\\ \ell(\lambda)\leq k}} q^{|\lambda|}$?

$\|$ rank generating function for $[\varnothing, k\times j]$ interval of Young's lattice $\mathbb{Y}$

e.g.



q^6

q^5

$2q^4$

$2q^3$

$2q^2$

q^1

$1\cdot q^0$

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6 = (1+q+q^2+q^3+q^4)(1+q^2)$

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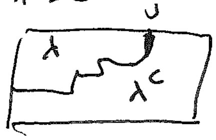
Let's collect some properties of $\begin{bmatrix} j+k \\ k \end{bmatrix}_q$ ...

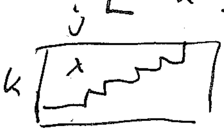
Prop (a) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q \xrightarrow{q=1} \binom{j+k}{k}$ (since $k\{$  gives lattice paths $(0,0) \to (j,k)$)

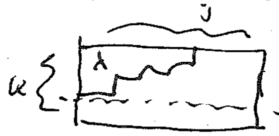
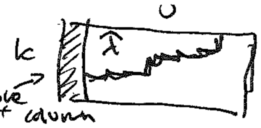
(b) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k \\ j \end{bmatrix}_q$ (since  $\longleftrightarrow$ )

(c) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q =: \sum_{n=0}^{jk} p(j,k,n) q^n$ has symmetric coefficients: $p(j,k,n) = p(j,k,jk-n)$,

(since  have $|\lambda| + |\lambda^c| = jk$) (e.g. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$ has coef's: $(1,1,2,2,2,1,1)$)

(d) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (1^{st} q-Pascal recurrence)

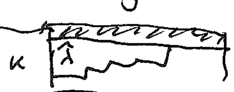
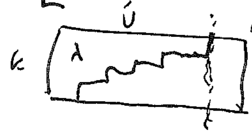


 or 

remove 1^{st} column

$= q^j \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$

remove 1^{st} row

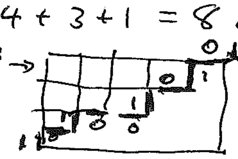
 or 

(e) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \sum_{\substack{\text{rearrangements} \\ (w_1,\dots,w_{j+k}) = w \text{ of } 0^j 1^k}} q^{inv(w)}$ where $inv(w) := \#\{(a,b) : 1 \le a < b \le j+k, \ w_a > w_b\}$ is # of inversions of w

$0^j 1^k = \underbrace{00\cdots0}_{j}, \underbrace{11\cdots1}_{k}$

e.g. $inv(01010010) = 4+3+1 = 8$

(Since can read boundary of λ backwards as 0 = west, 1 = south to get w and then $|\lambda| = inv(w)$

e.g. )

(f) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \#\{\underline{k\text{-dimensional subspaces}} \text{ of } (\mathbb{F}_q)^{j+k}\}$ if $q = p^d$ is a prime power, so $q = |\mathbb{F}_q|$

(g) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \dfrac{[j+k]!_q}{[j]!_q [k]!_q}$ where $[n]!_q := [1]_q [2]_q \cdots [n]_q$

$[n]_q := 1+q+q^2+\cdots+q^{n-1} = \dfrac{1-q^n}{1-q}$

e.g. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \dfrac{[5]!_q}{[3]!_q [2]!_q} = \dfrac{[5]_q [4]_q \cancel{[3]_q} \cancel{[2]_q} \cancel{[1]_q}}{\cancel{[3]_q} \cancel{[2]_q} \cancel{[1]_q} [2]_q [1]_q} = (1+q+q^2+q^3+q^4) \dfrac{(1+q+q^2+q^3)}{(1+q)}$

$= (1+q+q^2+q^3+q^4)(1+q^2)$ ✓

Pf: (a), (b), (c), (d), (e) proved in comments above
(we could prove (f) + (g) using (d) + induction, but we won't...)
For (f), we claim that there is a bijection:

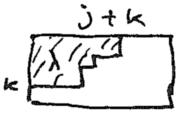
$$\{k\text{-dim'l subspaces } V \subseteq (\mathbb{F}_q)^{j+k}\} \quad \text{RowSpace}(A)$$

$\updownarrow$ see LEMMA below

$$\left\{ \begin{array}{l} \text{matrices } A \in \mathbb{F}_q^{k\times(j+k)} \text{ of (full)} \\ \text{rank } k \text{ in row-reduced echelon form} \end{array} \right\} \quad A$$

$\pi^{-1}(\lambda)$ has $|\pi^{-1}(\lambda)| = q^{|\lambda|}$

$\downarrow \pi$

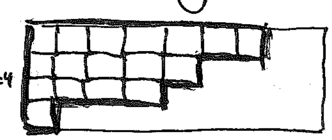
$\lambda \quad \{\text{partitions } \lambda \subseteq$  $\}$ shape of the $*$'s (= nonzero entries in non-pivot columns) read backwards

$j+k=13,\ j=9$

e.g. $k=4$
$$\begin{bmatrix} 0&0&1&*&*&0&*&0&*&*&*&0&* \\ 0&0&0&0&0&1&*&0&*&*&*&0&* \\ 0&0&0&0&0&0&0&1&*&*&*&0&* \\ 0&0&0&0&0&0&0&0&0&0&0&1&* \end{bmatrix}$$

$$\begin{array}{ccccccc} *&*&*&*&*&*&* \\ &&*&*&*&*&* \\ &&&*&*&*&* \\ &&&&&&* \end{array}$$

$\Updownarrow$



$k=4$, $j=9$

LEMMA: If $A, B \in \mathbb{F}_q^{k\times(j+k)}$ are both in RREF, and have same rowspace, then $A = B$.

Pf: Row Space $[-A-]$ = Row Space $[-B-]$ $\Leftrightarrow$
$PA = B$ for some $P \in GL_k(\mathbb{F}_q)$
think abt pivot columns $\Rightarrow P = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_{k\times k} \Rightarrow A = B$ ▨

Once you believe $|\pi^{-1}(\lambda)| = q^{|\lambda|}$ (can choose $*$'s from $\mathbb{F}_q$ arbitrarily), then
$$|\{k\text{-dim'l subspace } V \subseteq \mathbb{F}_q^{j+k}\}| = \sum_{\lambda \subseteq k\times j} |\pi^{-1}(\lambda)| \left(= \sum_{\lambda \subseteq k\times j} q^{|\lambda|}\right) = \begin{bmatrix} j+k \\ k \end{bmatrix}_q.$$

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For (g), it suffices to check $\#\{k\text{-dim'l subspaces } V \subseteq \mathbb{F}_q^{j+k}\} \overset{?}{=} \dfrac{[j+k]!_q}{[k]!_q [j]!_q}$ $\swarrow$ since $\exists$ infinitely many choices of $q = p^d$!

$$= \frac{\#\{\text{ordered bases } (v_1, v_2, \dots, v_k) \text{ for all } k\text{-dim'l subs. in } \mathbb{F}_q^{j+k}\}}{\#\{\text{ordered bases } (v_1, \dots, v_k) \text{ for one particular } k\text{-subspace}\}} =$$
like say $V = \mathbb{F}_q$

$$\frac{\overset{\text{pick } v_1}{(q^{j+k}-1)}\,\overset{\text{pick } v_2 \text{ not in } \mathbb{F}_q v_1}{(q^{j+k}-q)}\,(q^{j+k}-q^2)\cdots(q^{j+k}-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})} = \frac{(q^{j+k}-1)(q^{j+k-1}-1)(q^{j+k-2}-1)\cdots(q^{j+1}-1)}{(q^k-1)(q^{k-1}-1)(q^{k-2}-1)\cdots(q-1)}$$

$$= \frac{[j+k]_q [j+k-1]_q \cdots [j+1]_q}{[k]_q [k-1]_q \cdots [1]_q} = \frac{[j+k]!_q}{[k]!_q [j]!_q}.$$ ▨

More generally, one can define the <u>q-multinomial</u> <u>coefficient</u>

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q := \frac{[n]!_q}{[k_1]!_q [k_2]!_q \cdots [k_\ell]!_q} \quad \text{for } \sum_{i=1}^{\ell} k_i = n$$

$\ell = 2$, $(k_1, k_2) = (k, j)$

$$\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k \\ j \end{bmatrix}_q = \begin{bmatrix} j+k \\ k, j \end{bmatrix}_q \qquad \xrightarrow{q=1} \binom{n}{k_1, k_2, \ldots, k_\ell} \text{ usual multinomial}$$

<u>Prop</u>: (a) $\begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q = \sum_{\substack{\text{rearrangements} \\ w = (w_1, \ldots, w_n) \\ \text{of } k_1\ 1\text{'s}, \\ k_2\ 2\text{'s}, \ldots \\ k_\ell\ \ell\text{'s} \ldots}} q^{\mathrm{inv}(w)}$ $\left(\text{In particular, } \begin{bmatrix} n \\ 1, 1, \ldots, 1 \end{bmatrix}_q = [n]!_q = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} \right)$

(b) $\begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q = \#\left\{ \begin{array}{c} \text{partial flags of subspaces} \\ \{0\} \subset V_{k_1} \subset V_{k_1+k_2} \subset \cdots \subset V_{k_1+k_2+\cdots+k_{\ell-1}} \subset \mathbb{F}_q^n, \\ \text{w/ } \dim_{\mathbb{F}_q} V_i = i \end{array} \right\}$

(In particular $[n]!_q = \#\{\text{complete flags } \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{F}_q^n\}$).

<u>Proof</u>: For both, use

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q \overset{\text{easy!}}{=} \underbrace{\begin{bmatrix} n \\ k_1 \end{bmatrix}_q}_{q\text{-binomial}} \cdot \underbrace{\begin{bmatrix} n-k_1 \\ k_2, k_3, \ldots, k_\ell \end{bmatrix}_q}_{\text{smaller } q\text{-multinomial}}$$

(Base cases $\ell = 1 \Rightarrow$ trivial, $\ell = 2 \Rightarrow$ already done in previous prop.)

and in the inductive step:

- for (a), note that $\mathrm{inv}(w) = \#\{\text{inversions between 1's and all of 2's, 3's, } \ldots, \ell\text{'s}\}$ $+ \#\{\text{inversions between 2's, 3's, } \ldots, \ell\text{'s}\}$

e.g. $w = 124213241$

$\mathrm{inv}(w) = \mathrm{inv}(122212221)$
$+ \mathrm{inv}(242\ \ 324)$ ✓

- for (b), note that after fixing V_{k_1},

$$\{\text{flags } \{0\} \subset V_{k_1} \subset V_{k_1+k_2} \subset \cdots \subset \mathbb{F}_q^n\} \leftrightarrow \{\text{flags } \{0\} \subset V_{k_1+k_2}/V_{k_1} \subset V_{k_1+k_2+k_3}/V_{k_1} \subset \cdots \subset \underbrace{\mathbb{F}_q^n/V_{k_1}}_{\cong\, \mathbb{F}_q^{n-k_1}}\}.$$

Geometry/topology digression

For any field $\mathbb{F}$ (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{F}_q, \ldots$) one defines $\xrightarrow{k=1}$ $\mathbb{P}^{n-1}_{\mathbb{F}} := \{$projective space of lines in $\mathbb{F}^n\}$ $[n]_q$

$\begin{bmatrix} n \\ k \end{bmatrix}_q$ $Gr(k, \mathbb{F}^n) := \{$Grassmannian of k-dim'l $\mathbb{F}$-subspaces in $\mathbb{F}^n\}$

$[n]!_q$ $Fl(n) := \{$flag manifold complete flags $\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{F}^n\}$

$\begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q$ $Fl_{k_1,\ldots,k_\ell}(n) := \{$partial flag manifold of partial flags $\{0\} \subset V_{k_1} \subset \cdots \subset V_{k_1+\cdots+k_{\ell-1}} \subset \mathbb{F}^n\}$

and they turn out to be smooth projective varieties $\forall\ \mathbb{F}$, and (smooth) manifolds for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, with a Schubert/Bruhat cell decomposition for

$$Fl_{k_1,\ldots,k_\ell}(n) = \bigsqcup_{\substack{\text{rearrangements} \\ w \text{ of } 1^{k_1}, 2^{k_2}, \ldots, \ell^{k_\ell}}} X_w \quad \text{with } X_w \cong \mathbb{F}^{\mathrm{inv}(w)}$$

a cell (i.e. "open ball") of dimension $\mathrm{inv}(w)$

whose closures $\overline{X_w}$ are called Schubert (sub-)varieties.

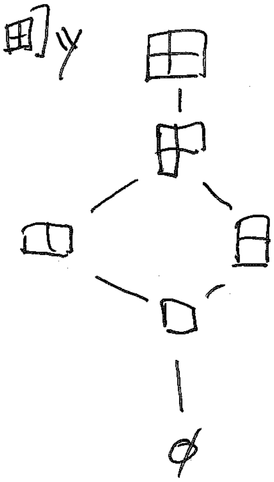
They help not only count $|Fl_{k_1,\ldots,k_\ell}(n)| = \begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q$ for $\mathbb{F} = \mathbb{F}_q$, but also compute the (co-)homology for $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

The poset of cells $(X_w, \leq_{\text{Bruhat}})$ ordered by $X_w \leq X_{w'}$ if $\overline{X_w} \subseteq \overline{X_{w'}}$ is Bruhat order.

For $Gr(k, \mathbb{F}^n)$ this poset is $[\varnothing, k \times (n-k) \text{ box}] \subset Y$:

e.g. $k=2, n=4$

$[\varnothing, ⊞] \subset Y$



$$\left\{\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}\right\}$$

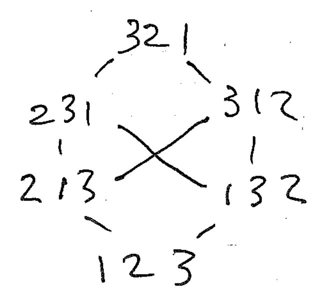
$$\left\{\begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\} \qquad \left\{\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}$$

Bruhat order on $Fl_n \Leftrightarrow (\mathfrak{S}_n, \leq_{\text{Bruhat}})$ where $\leq_{\text{Bruhat}}$ is the transitive closure of $x < y$ if $y = x(i,j)$ for some i,j and $\mathrm{inv}(y) = \mathrm{inv}(x)+1$

e.g. $\mathfrak{S}_3$



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Descents of permutations

DEF'N For $w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix} \in \mathfrak{S}_n$,

its descent set $D(w) := \{i : 1 \le i \le n-1,\ w_i > w_{i+1}\}$

$\mathrm{des}(w) := |D(w)|$ descent number

$\mathrm{maj}(w) := \sum_{i \in D(w)} i$ major index (considered by MacMahon)

Eulerian polynomial $A_n(x) := \sum_{w \in \mathfrak{S}_n} x^{1+\mathrm{des}(w)}$

Mahonian polynomial $\mathrm{Mahon}(q) := \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)}$.

EX $n=1$: $A_1(x) = x^1 = x$

$\mathrm{Mahon}(q) = q^0 = 1 = [1]!_q$

$n=2$: $A_2(x) = \underset{12}{x^1} + \underset{21}{x^2}$

$\mathrm{Mahon}(q) = q^0 + q^1 = 1 + q = [2]!_q$

$n=3$

w	$\mathrm{des}(w)$	$\mathrm{maj}(w)$
123	0	0
132	1	2
213	1	1
231	1	2
312	1	1
321	2	3

$A_3(x) = x + 4x^2 + x^3$

$\mathrm{Mahon}(q) = 1 + 2q + 2q^2 + q^3$
$= (1+q)(1+q+q^2)$
$= [3]!_q$

$n=4$ $A_4(x) = x + 11x^2 + 11x^3 + x^4$, $\mathrm{Mahon}(q) = [4]!_q$

THM 1 $\mathrm{Mahon}(q) = [n]!_q$,

i.e. $\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} = [n]!_q = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)}$

Stanley proves this bijectively in §1.4

∃ bijective proof of this using codes of permutations

$$(x \cdot d/dx)^n \left(\frac{1}{1-x}\right)$$ ← The way Euler thought about these numbers

$=$

<u>THM 2</u>: $\sum_{m \geq 0} m^n x^m \doteq \frac{A_n(x)}{(1-x)^{n+1}}$ $(*)$

and consequently, $\sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}$ $(**)$

(Why does $(*) \Rightarrow (**)$? $(*)$ gives $\sum_{n \geq 0} \frac{A_n(x)}{(1-x)^{n+1}} \frac{t^n}{n!} = \sum_{n \geq 0, m \geq 0} x^m \frac{m^n t^n}{n!}$

$$= \sum_{m \geq 0} \underbrace{x^m e^{mt}}_{(xe^t)^m} = \frac{1}{1-xe^t}$$

$$\Rightarrow \sum_{n \geq 0} A_n(x) \frac{(t/(1-x))^n}{n!} = \frac{1-x}{1-xe^t}$$

$\big\}$ replace t by $t(1-x)$

$$\sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}.$$

)

10/21 Let's deduce these from...

<u>THM</u> (a) $\left(\frac{1}{1-q}\right)^n = \frac{\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}$ ($\Rightarrow$ THM 1 by clearing denominator)

(b) $\sum_{m \geq 0} ([m]_q)^n x^m = \frac{\sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)}}{(1-x)(1-xq)(1-xq^2)\cdots(1-xq^n)}$ ($\Rightarrow$ THM 2 by $\lim q \to 1$)

<u>Proof</u>: For (a), note that

$$\mathrm{LHS} = \left(\frac{1}{1-q}\right)^n = \sum_{\substack{f:[n] \to \mathbb{N} \\ \shortparallel \\ (f_1, f_2, \ldots, f_n)}} q^{\overbrace{f_1 + f_2 + \cdots + f_n}^{|f|}}$$

(simple)

<u>LEMMA</u>: Every $f:[n] \to \mathbb{N}$ has a unique permutation $w \in \mathfrak{S}_n$ such that f is <u>w-compatible</u> in the sense that

- $f_{w_1} \geq f_{w_2} \geq \cdots \geq f_{w_n}$
- and $f_{w_i} > f_{w_{i+1}}$ if $i \in D(w)$ (i.e., $w_i > w_{i+1}$)

Pf of lemma:

e.g. $f=(\overset{1}{2},\overset{2}{0},\overset{3}{5},\overset{4}{0},\overset{5}{3},\overset{6}{3},\overset{7}{2},\overset{8}{0})$ has $\overset{5}{f_3}\geq \overset{3}{f_5}\geq \overset{3}{f_6} > \overset{2}{f_1}\geq \overset{2}{f_7} > \overset{0}{f_2}\geq \overset{0}{f_4}\geq \overset{0}{f_8}$

descents

so is w-compatible for $w=(3,5,6,\cdot 1,7\cdot 2,4,8)\in\mathfrak{S}_8$. ∎

Thus $LHS = \sum_{w\in\mathfrak{S}_n}\sum_{\substack{f:[n]\to\mathbb{N}\\ w\text{-compatible}}} q^{|f|}$

Subtract ~~off~~ the smallest w-compatible f_0 from f to get λ:

$$\begin{array}{rl} & \overset{3}{5},\overset{5}{3},\overset{6}{3},\overset{\cdot 1}{2},\overset{7}{2},\overset{\cdot 2}{0},\overset{4}{0},\overset{8}{0} \\ & (5,3,3,2,2,0,0,0)=f \\ - & (2,2,2,1,1,0,0,0)=f_0 \\ \hline & (3,1,1,1,1,0,0,0)=\lambda \end{array}$$

(NOTE: $|f_0| = \mathrm{maj}(w)$)

(and $\max(f_0) = \mathrm{des}(w)+1$)

$= \sum_{w\in\mathfrak{S}_n}\sum_{\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)} q^{\mathrm{maj}(w)+|\lambda|}$

$= \sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} \sum_{\lambda:\ell(\lambda)\leq n} q^{|\lambda|}$

$= \sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} \dfrac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$ ✓

For (b), we'll do same thing similar, showing

$$(1-x)\sum_{m\geq 0}([m]_q)^n x^m = \frac{\sum_{w\in\mathfrak{S}_n}' x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)}}{(1-xq)(1-xq^2)\cdots(1-xq^n)}$$

Note/ $LHS = (1-x)\sum_{m\geq 0}x^m \sum_{f:[n]\to\{0,1,\ldots,m-1\}} q^{|f|} = (1-x)\sum_{m\geq 0}x^m\sum_{\substack{f:[n]\to\mathbb{N}\\ \max(f)\leq m-1}} q^{|f|}$

cancellation from $(1-x)$ factor

$= \sum_{m\geq 0} x^m \sum_{\substack{f:[n]\to\mathbb{N}\\ \max(f)=m-1}} q^{|f|} = \sum_{f:[n]\to\mathbb{N}} x^{\max(f)+1} q^{|f|}$

$= \sum_{w\in\mathfrak{S}_n}\sum_{\substack{f:[n]\to\mathbb{N}\\ w\text{-compatible}}} x^{\max(f)+1} q^{|f|}$

Subtract off the smallest w-compatible f_0 from f to get λ

$= \sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)} \underbrace{\sum_{\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)}}\ x^{\overset{\lambda_1}{\max(\lambda)}} q^{|\lambda|}$ same as $\sum_{\lambda:\lambda_1\leq n} x^{\ell(\lambda)} q^{|\lambda|}$ via $\lambda \leftrightarrow \lambda^t$

$= \sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)} \cdot \dfrac{1}{(1-xq)(1-xq^2)(1-xq^3)\cdots(1-xq^n)}$ ✓

∎

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REMARKS

① $\sum_{w\in \mathfrak{S}_n} x^{des(w)} = \sum_{w\in \mathfrak{S}_n} x^{asc(w)}$ where $asc(w) := \#\text{ascents of } w$
$= \#\{1\le i\le n-1 : w_i < w_{i+1}\}$
$= (n-1) - des(w)$

and they have symmetric coefficient sequence
(e.g. $\cancel{A_4(x)}$ $1+11x+11x^2+x^3 = \sum_{w\in\mathfrak{S}_4} x^{des(w)}$)

since $des(\overset{w}{w_1 w_2 \cdots w_n}) = asc(\overset{w_0 w}{(n+1-w_1, n+1-w_2, \ldots, n+1-w_n)}) = asc(\overset{w w_0}{(w_n, w_{n-1}, \ldots, w_2, w_1)})$,

where $w_0 = \begin{pmatrix}1 & 2 & \cdots & n-1 & n\\ n & n-1 & \cdots & 2 & 1\end{pmatrix} \in \mathfrak{S}_n$ (the so-called "longest word").

② The map $w \longmapsto \hat{w}$ that sent $\#cyc(w) = \#\text{L-to-R-max}(\hat{w})$

$\overset{\shortparallel}{(2)(7\,1\,6)(8)(9\,4\,3\,5)}$ $\quad \overset{\shortparallel}{2\,7\,1\,6\,8\,9\,4\,3\,5}$

has the property that $1 + asc(\hat{w}) = \#\{1\le i\le n : \underbrace{i\le w(i)}_{\text{called a weak exceedance of } w}\}$

$n-(-)$ {

$des(\hat{w}) = n - \#\{1\le i\le n : i\le w(i)\}$
$= \#\{1\le i\le n : \underbrace{i > w(i)}_{\text{called a non-exceedance of } w}\}$

Hence $\sum_{w\in\mathfrak{S}_n} x^{des(w)} = \sum_{w\in\mathfrak{S}_n} x^{\#\text{non-exc}(w)}$
$= \sum_{w\in\mathfrak{S}_n} x^{\#exc(w)}$

where $exc(w) = \#\{1\le i\le n : w(i) > i$

e.g. $n=3$

w	$exc(w)$	$des(w)$
$\binom{1\,2\,3}{1\,2\,3}$	0	0
$\binom{1\,2\,3}{1\,3\,2}$	1	1
$\binom{1\,2\,3}{2\,1\,3}$	1	1
$\binom{1\,2\,3}{2\,3\,1}$	2	1
$\binom{1\,2\,3}{3\,1\,2}$	1	1
$\binom{1\,2\,3}{3\,2\,1}$	1	2

Q: Can we count $\beta(S) := \#\{w\in\mathfrak{S}_n : D(w) = S\}$?
for a subset $S \subseteq [n-1]$ $\quad q=1$

Or even better, $\beta(S,q) := \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w)=S}} q^{inv(w)}$?

e.g. $n=4$, $S=\{2\}$

$w : D(w)=\{2\}$	$inv(w)$
$13\cdot 24$	1
$14\cdot 23$	2
$23\cdot 14$	2
$24\cdot 13$	3
$34\cdot 12$	4

$\Rightarrow \beta(S,q) = q+2q^2+q^3+q^4$
$\quad\downarrow q=1$
$\beta(S) = 5$

It turns out to be easier to count $\alpha(S) := \#\{w : D(w) \subseteq S\}$

and $\alpha(S,q) := \sum_{\substack{w \in \mathfrak{S}_n \\ D(w) \subseteq S}} q^{inv(w)}$ $\xleftarrow{q=1}$

since $inv(w^{-1}) = inv(w)$

$$= \sum_{\substack{w \in \mathfrak{S}_n: \\ D(w^{-1}) \subseteq S}} q^{inv(w)}$$

$$= \sum_{\substack{\text{rearrangements} \\ w = (w_1, w_2, \ldots, w_n) \\ \text{of } 1^{k_1}, 2^{k_2}, \ldots, \ell^{k_\ell}}} q^{inv(w)} = \begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q$$

where $\underline{k} = (k_1, k_2, \ldots, k_\ell) \models n$ is the composition for which

$S =$ partial sums $\{k_1, k_1+k_2, \ldots, k_1+k_2+\cdots+k_\ell\} \subseteq [n-1]$

because $\{w \in \mathfrak{S}_n : D(w^{-1}) \subseteq S\} =$ shuffles of $1 < 2 < \cdots < k_1$

$k_1+1 < k_1+2 < \cdots < k_1+k_2$

$\vdots$

$k_1+\cdots+k_{\ell-1}+1 < \cdots < k_1+k_2+\cdots+k_n$

inverse descents can only occur here

e.g. $S = \{3, 5\} \subseteq [\overset{n}{\overset{\shortparallel}{8}}-1]$

$\Updownarrow$

$\underline{k} = (3,2,3) \models 8$

rearrangement		shuffle
111 22 333	$\leftrightarrow$	123 456 78
231 33 211	$\leftrightarrow$	461 78 523

10/25 So how do we recover $\beta(S)$, $\beta(S,q)$ from $\alpha(S) = \sum_{T \subseteq S} \beta(S)$? $\alpha(S,q)$

Prop (Principle of Inclusion-Exclusion)

Given two functions $f_{\leq}, f_{=} : 2^{[n]} \to R$ ← any abelian group

$S \longmapsto f_{\leq}(S)$

$S \longmapsto f_{=}(S)$

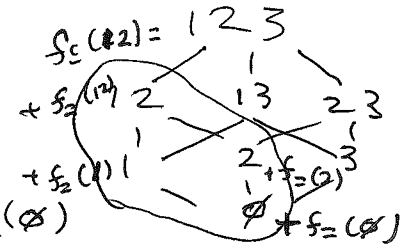
then $f_{\leq}(S) \overset{(*)}{=} \sum_{T \subseteq S} f_{=}(T) \quad \forall S \subseteq [n]$

$$\Longleftrightarrow f_{=}(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_{\leq}(T)$$

e.g. $f_=(\emptyset) = f_\subseteq(\emptyset)$

$$f_=(\{i\}) = f_\subseteq(\{i\}) - f_\subseteq(\emptyset)$$

$$f_=(\{i,j\}) = f_\subseteq(\{i,j\}) - f_\subseteq(\{i\}) - f_\subseteq(\{j\}) + f_\subseteq(\emptyset)$$



<u>Cor</u> Let $f_\subseteq(S) := \alpha(S,q) = \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w^{-1})\subseteq S}} q^{inv(w)} = \begin{bmatrix} n \\ k_1,\ldots,k_\ell\end{bmatrix}_q$

Then $f_=(S) = \beta(S,q) = \sum_{\substack{w\in\mathfrak{S}_n\\ D(w^{-1})\subseteq S}} q^{inv(w)} = \sum_{T\subseteq S}\alpha(T,q)(-1)^{|S\setminus T|}$

$$= \sum_{\substack{\underline{k}'\models n,\\ \text{coarsening } \underline{k}}} (-1)^{\ell(\underline{k})-\ell(\underline{k}')} \begin{bmatrix} n\\ \underline{k}'\end{bmatrix}_q.$$

<u>e.g.</u> $n=4$, $S=\{2\}$, $\underline{k}=(2,2)$

$$\beta(\{2\},q) = \alpha(\{2\},q) - \alpha(\emptyset,q)$$
$$= \begin{bmatrix} 4\\ 2,2\end{bmatrix}_q - \begin{bmatrix} 4\\ 4\end{bmatrix}_q = \frac{[4]_q[3]_q}{[2]_q} - 1$$
$$= (1+q^2)(1+q+q^2) - 1 = \cancel{1}+q+2q^2+q^3+q^4\cancel{-1}$$
$$= q+2q^2+q^3+q^4 \checkmark$$

<u>Proof of PIE</u>: Note $\{f_=(S)\}_{S\subseteq[n]}$ determines $\{f_\subseteq(S)\}_{S\subseteq[n]}$ <u>uniquely</u> via $(*)$, and conversely <u>by induction on $|S|$</u>, since $(*)$ says

$$f_=(S) = f_\subseteq(S) - \underbrace{\sum_{T\subsetneq S} f_=(T)}_{\text{already determined}}$$

If we define $g(R) := \sum_{T\subseteq R}(-1)^{|R\setminus T|} f_\subseteq(T)\ \forall R\subseteq[n]$,

Then fixing some $S\subseteq[n]$, $\sum_{R\subseteq S} g(R) = \sum_{R\subseteq S}\sum_{T\subseteq R}(-1)^{|R\setminus T|} f_\subseteq(T)$

$$= \sum_{T\subseteq S} f_\subseteq(T) \underbrace{\sum_{T\subseteq R\subseteq S}(-1)^{|R\setminus T|}}$$

$$= \sum_{\hat{R} := R\setminus T\subseteq S\setminus T} (-1)^{|\hat{R}|} = \begin{cases} 1 & \text{if } S=T\\ \sum_{k=0}^{|S\setminus T|=m}(-1)^k\binom{m}{k} = (1+(-1))^m = 0 & \text{if } T\subsetneq S.\end{cases}$$

$\Downarrow$

$$g(S) = f_\subseteq(S) - \sum_{T\subsetneq S} g(T)$$

$\Downarrow$

$$g(S) = f_=(S)\ \forall S \checkmark$$

$\boxtimes$

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Examples of PIE

① Determinantal reformulation

Prop: If it happens that $f_{\subseteq}(S) = h(n)e(k_1)e(k_2)\cdots e(k_\ell)$
when $S \subseteq [n-1]$ = partial sums of $\underline{k} = (k_1, \dots, k_\ell) \models n$
for some $h, e : \mathbb{Z} \longrightarrow R$ ← a commutative ring,

$$\text{then } f_=(S) = h(n)\cdot \det \begin{bmatrix} e(k_1) & e(k_1+k_2) & e(k_1+k_2+k_3) & \cdots \\ 1 & e(k_2) & e(k_2+k_3) & \cdots \\ 0 & 1 & \ddots & \ddots \\ 0 & 0 & 1 & e(k_\ell) \end{bmatrix}$$

e.g. $n=9$

$$f_=(\{3,5\}) = h(9)\det\begin{bmatrix} e(3) & e(5) & e(9) \\ 1 & e(2) & e(6) \\ 0 & 1 & e(4) \end{bmatrix} = h(9)\begin{pmatrix} e(3)e(2)e(4) \\ - e(5)e(4) - e(3)e(6) \\ + e(9) \end{pmatrix}$$

$\updownarrow$ $\underline{k} = (3,2,4)$

COR $\alpha(S,q) = \begin{bmatrix} n \\ k_1, \dots, k_\ell \end{bmatrix}_q = \underbrace{[n]!_q}_{h(n)} \cdot \underbrace{\frac{1}{[k_1]!_q}}_{e(k_1)} \cdots \underbrace{\frac{1}{[k_\ell]!_q}}_{e(k_\ell)}$

$$\text{so } \beta(S,q) = [n]!_q \cdot \det \begin{bmatrix} \frac{1}{[k_1]!_q} & \frac{1}{[k_1+k_2]!} & \cdots & \frac{1}{[n]!} \\ 1 & \frac{1}{[k_2]!_q} & \cdots & \\ & 1 & \ddots & \\ 0 & & 1 & \frac{1}{[k_\ell]!_q} \end{bmatrix}$$

e.g. $n=4$

$$\beta(\{2\}, q) = [4]!_q \begin{bmatrix} \frac{1}{[2]!_q} & \frac{1}{[4]!_q} \\ 1 & \frac{1}{[2]!_q} \end{bmatrix} = [4][3][2]\left(\frac{1}{[2][2]} - \frac{1}{[4][3][2]}\right) = (1+q^2)(1+q+q^2) - 1 \checkmark$$

$\updownarrow$ $\underline{k} = (2,2)$

② Similarly, if $f_{\supseteq}(S) = \sum_{T \supseteq S} f_=(T)$

then $f_=(S) = \sum_{T \supseteq S} (-1)^{|T \setminus S|} f_{\supseteq}(T)$

and in particular, $f_=(\emptyset) = \sum_T (-1)^{T} f_{\supseteq}(T)$.

e.g. if $A_1, A_2, \dots, A_n$ are subsets of some universe U,

then letting $f_{\supseteq}(S) = \#(\bigcap_{i \in S} A_i) = \#\{u \in U : \{i=1,\dots,n : u \in A_i\} \supseteq S\}$

common formulation of PIE.

then $f_=(S) = \#\{u \in U : \{i = 1, \dots, n : u \in A_i\} = S\}$

$$= \sum_{T \supseteq S} (-1)^{|T \setminus S|} \#\left(\bigcap_{i \in T} A_i\right),$$ and in particular,

$$\boxed{\#\left(U \setminus \left(\bigcup_{i=1}^{n} A_i\right)\right) = f_=(\emptyset) = \sum (-1)^{|T|} \#\left(\bigcap_{i \in T} A_i\right) = |U| - \sum_{i=1}^{n} \#A_i + \sum_{1 \le i < j \le n} \#A_i \cap A_j - \cdots}$$

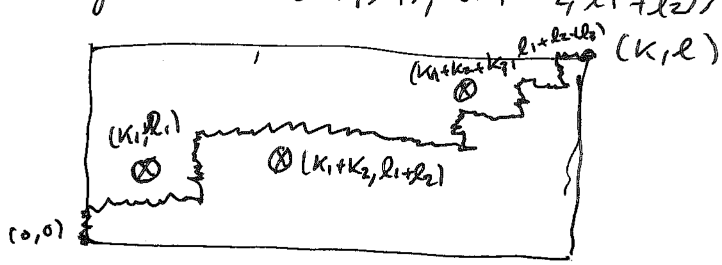
e.g. $d_n = \#\{\text{derangements } \sigma \in \mathfrak{S}_n\} = \#\left(U \setminus \bigcup_{i=1}^{n} A_i\right)$ w/ $A_i = \{\sigma \in \mathfrak{S}_n : \sigma(i) = i\}$

$$= \sum_{T \subseteq [n]} (-1)^{|T|} \#\left(\bigcap_{i \in T} A_i\right) \qquad \#\left(\bigcap_{i \in T} A_i\right) = \#\{\sigma \in \mathfrak{S}_n : \sigma(i) = i \ \forall i \in T\} = (n - |T|)!$$

$$= \sum_{T \subseteq [n]} (-1)^{|T|} (n - |T|)! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) \checkmark$$

③ How many lattice paths $(0,0) \to (k, l)$, $k = k_1 + k_2 + \cdots + k_m$, $l = l_1 + l_2 + \cdots + l_m$, avoid the points $(k_1, l_1), (k_1 + k_2, l_1 + l_2), \dots, (k_1 + k_2 + \cdots + k_{m-1}, l_1 + \cdots + l_{m-1})$?



if $A_i = \{\text{paths that hit pt } (k_1 + \cdots + k_i, l_1 + \cdots + l_i)\}$

then $\#A_i = \binom{k_1 + \cdots + k_i + l_1 + \cdots + l_i}{k_1 + \cdots + k_i}\binom{k_{i+1} + \cdots + k_m + l_{i+1} + \cdots + l_m}{k_{i+1} + \cdots + k_m}$

$\#A_i \cap A_j = (\cdots) \cdot (\cdots) \cdot (\cdots)$

and $\#\left(U \setminus \left(\bigcup_{i=1}^{m} A_i\right)\right) = \sum_T (-1)^{|T|} \# \bigcap_{i \in T} A_i$

$$= \det \begin{bmatrix} \binom{k_1+l_1}{k_1} & \binom{k_1+k_2+l_1+l_2}{k_1+k_2} & \cdots & \binom{k+l}{k} \\ 1 & \binom{k_2+l_2}{k_2} & \cdots & \\ 0 & \ddots & \ddots & \\ 0 & 0 & 1 & \binom{k_{m-1}+l_{m-1}}{k_{m-1}} \end{bmatrix}$$

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