0130

Sign-reversing involutions + identities involving signs

Some identities w/ +/- signs can be proven like this;

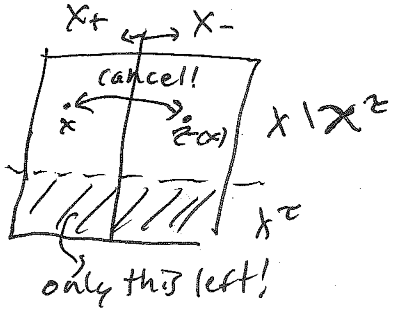
Prop Given a set X with a sign function $\mathrm{sgn}: X \to \{\pm 1\}$
a weight function $wt: X \to R$ ← abelian group

and a sign-reversing ($\mathrm{sgn}(\tau(x)) = -\mathrm{sgn}(x)$ if $\tau(x) \neq x$), weight-preserving ($wt(\tau(x)) = wt(x)$), involution ($\tau^2 = id$)

$\tau: X \to X$,

then $$\sum_{x \in X} \mathrm{sgn}(x) \cdot wt(x) = \sum_{x \in X^\tau := \{x \in X : \tau(x) = x\}} \mathrm{sgn}(x) \cdot ~~wt~~ t(x).$$

Proof: $X =$



$$\mathrm{sgn}(x) wt(x) + \underbrace{\mathrm{sgn}(\tau(x))}_{-\mathrm{sgn}(x)} \underbrace{wt(\tau(x))}_{wt(x)} = 0$$
for $x \in X \setminus X^\tau$. $\square$

Examples (continued for many pages ...)

① (Warm-up) $$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0 \quad \text{for } n \geq 1$$

$$\overset{\shortparallel}{\sum_{\text{subsets } S \subseteq [n]} (-1)^{|S|}}$$

$\mathrm{sgn}: X = 2^{[n]} \to \{\pm 1\}$
$S \mapsto (-1)^{|S|}$

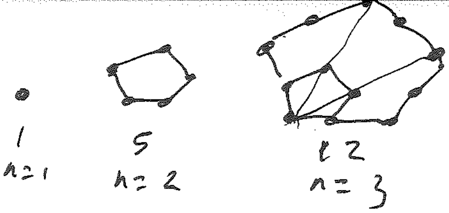
$wt: X = 2^{[n]} \to \mathbb{N}$
$S \mapsto 1$

$\tau: \overset{2^{[n]}}{\overset{\shortparallel}{X}} \to X$

$$S \mapsto \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$$

is sign-reversing, weight-preserving, with $X^\tau = \emptyset$ (no fixed pts).

pentagonal numbers:



② Recall **Thm** (Euler's "Pentagonal Number Theorem")

$$\underbrace{\prod_{j\geq 1}(1-q^j)}_{\text{denominator of the } p(n):=\#\{\lambda\vdash n\} \text{ generating function}} = 1 + \sum_{n\geq 1}(-1)\left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}}\right)$$

partition number

$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

Recall ⟹ **Cor** For $n\geq 1$,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

pf: $\displaystyle\sum_{n\geq 0} p(n)q^n = \frac{1}{\prod_{j\geq 1}(1-q^j)} \Rightarrow \left(\sum_{n\geq 0} p(n)q^n\right)\left(\underbrace{\prod_{j\geq 1}(1-q^j)}_{(1-q^1-q^2+q^5+q^7+\cdots)} = 1\right)$

$[q^n]$

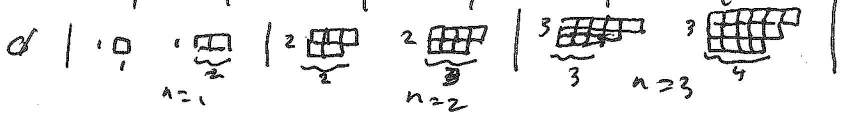
$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots = 0 \quad ▨$$

Franklin's (1881) proof of Euler's P.N.T.:

$$\text{LHS} = \prod_{j\geq 0}(1-q^j) = \sum_{\substack{\lambda:\ \lambda \text{ has distinct} \\ \text{parts } \lambda_1>\lambda_2>\cdots>\lambda_\ell}} \overbrace{(-1)^{\ell(\lambda)}}^{\operatorname{sgn}(x)} \overbrace{q^{|\lambda|}}^{\operatorname{wt}(x)}$$

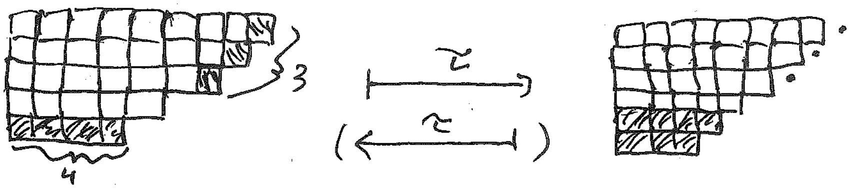
$\underline{X :=}$

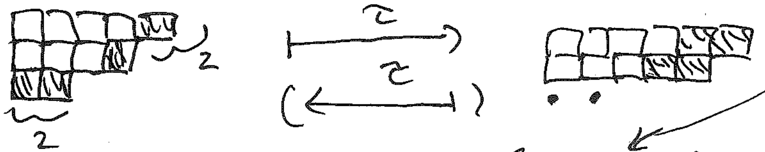
$$\text{RHS} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots -$$



Franklin defined $\tau : X = \{\lambda \text{ w/ distinct parts}\} \to X$ by comparing
- smallest part and
- longest initial run $\lambda_1, \lambda_1 - 1, \lambda_1 - 2, \ldots$

and moving the smaller one onto the bigger: (or smallest part onto longest run if they are the same size)



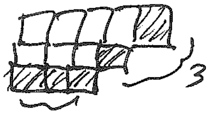


involution sign-reversing wt-preserving

when one can do this, check $\tau^2 = 1$, $\ell(\tau(\lambda)) = \ell(\lambda) \pm 1$, $|\lambda| = |\tau(\lambda)|$

One cannot do this:

- if they have the same size and overlap



$n=3$, $|\lambda| = \frac{3n(n-1)}{2}$

- or if the run is 1 smaller and they overlap

$|\lambda| = \frac{3n(n+1)}{2}$

So sign-reversing involution $\Rightarrow$ Only these shapes contribute to LHS $\Rightarrow$ LHS = RHS ▨

③ Theorem (Kirchoff's Matrix-Tree Theorem)

$[n] = \{1, \dots, n\}$

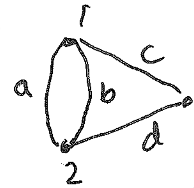
The number of spanning trees in a multigraph $G = (V, E)$ (~~multiple~~ parallel edges allowed!)

is $\det(\overline{L(G)}^{i,i})$, where $\overline{L(G)}^{i,i} = L(G)$ w/ row i, column i removed, for any $i=1,\dots,n$,

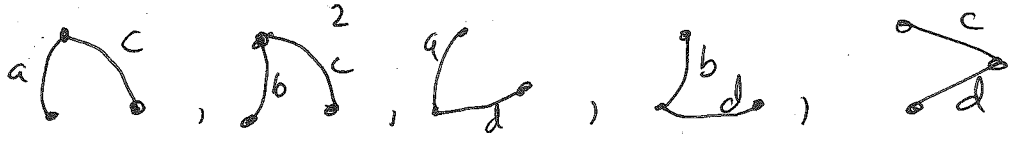
and $L(G)$ is $L(G)_{v,w} := \begin{cases} \deg_G(v) & \text{if } v=w \\ -\#\text{edges from } v \text{ to } w & \text{if } v \neq w \end{cases}$

$n \times n$ Laplacian matrix

Example $G =$



has 5 spanning trees:



and $L(G) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3 & -2 & -1 \\ 2 & -2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{array} \longrightarrow \det(\overline{L(G)}^{1,1})$

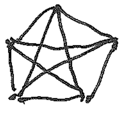
$= \det\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = 6 - 1 = 5$ ✓

$\det(\overline{L(G)}^{3,3}) = \det\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

$= 9 - 4 = 5$ ✓

Ex: Let's prove Cayley's formula n^{n-2} for spanning trees in complete graph K_n on $[n]$ this way...

e.g. $n=5$



$$\overline{L(K_n)}^{n,n} = \begin{array}{c|cccc|c} & 1 & 2 & \cdots & & n \\ \hline 1 & n-1 & -1 & \cdots & -1 & -1 \\ 2 & -1 & n-1 & & & \vdots \\ \vdots & \vdots & & \ddots & & -1 \\ \hline n & -1 & -1 & \cdots & -1 & n-1 \end{array} = n \underbrace{I_{n-1}}_{(n-1)\times(n-1) \text{ identity matrix}} - \underbrace{\mathbb{1}_{n-1}}_{\text{all 1's matrix } \left[\begin{smallmatrix}1 & \cdots & 1\\ \vdots & & \vdots \\ 1 & \cdots & 1\end{smallmatrix}\right]}$$

Who are eigenvalues of $\mathbb{1}_{n-1}$? It has rank 1, so $\underline{(n-2) \text{ eigenvalues} = 0}$

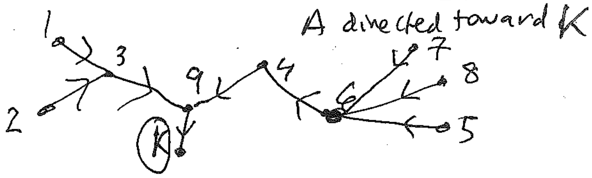
Also $\mathbb{1}_{n-1} \cdot \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = (n-1)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$ so $\underline{\text{one eigenvalue is } n-1}$.

Thus $\mathbb{1}_{n-1}$ has eigenvalues $(\underbrace{0,0,\ldots,0}_{n-2}, n-1)$, so $\overline{L(K)}^{n,n}$ has eigenvalues $(\underbrace{n,n,\ldots,n}_{n-2}, 1) \Rightarrow \det = n^{n-2}$. ✓

Instead of proving Kirchoff's Thm, let's prove a <u>weighted</u>, <u>directed</u> version:

<u>Thm</u> If
$$L = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} a_{12}+a_{13}+\cdots+a_{1n} & -a_{12} & -a_{13} \cdots & -a_{1n} \\ -a_{21} & a_{21}+a_{23}+\cdots+a_{2n} & -a_{23} & \cdots \\ -a_{31} & & & \\ \vdots & & & \\ -a_{n1} & \cdots & & a_{n1}+a_{n2}+\cdots+a_{n,n-1} \end{bmatrix}\end{array}$$
has $L_{ij} = \begin{cases} a_{i1}+a_{i2}+\cdots+\widehat{a_{ii}}+\cdots+a_{i,n} & \text{if } i=j \\ -a_{ij} & \text{if } i \neq j \end{cases}$

then $\det(\overline{L}^{k,k}) = \sum\limits_{\substack{\text{aborescences} \\ A \text{ directed toward } k}} \prod\limits_{\text{arcs } i\to j \text{ in } A} a_{ij} \in \mathbb{Z}[a_{12}, a_{21}, \ldots]$



Note: $\Rightarrow$ Kirchoff's Thm by setting $a_{ij} := \#(\text{edges } i \text{ to } j \text{ in } G) = a_{ji}$

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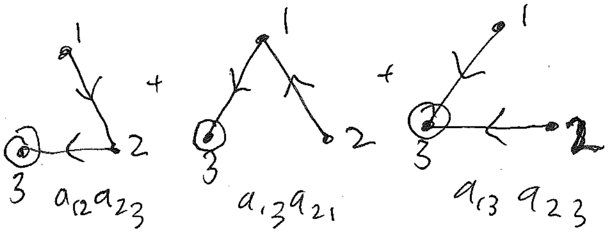
e.g. $n=3$
$$L = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & a_{12}+a_{13} & -a_{12} & -a_{13} \\ 2 & -a_{21} & a_{21}+a_{23} & -a_{23} \\ 3 & -a_{31} & -a_{32} & a_{31}+a_{32} \end{array} \Rightarrow \det(\overline{L}^{3,3}) = \det\begin{bmatrix} a_{12}+a_{13} & -a_{12} \\ -a_{21} & a_{21}+a_{23} \end{bmatrix}$$

$$= (a_{12}+a_{13})(a_{21}+a_{23}) - (-a_{12})(-a_{21})$$

$$= a_{12}a_{23} + \cancel{a_{12}a_{21}} + a_{13}a_{21} + a_{13}a_{23} - \cancel{a_{12}a_{21}}$$



$= a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{23}$

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Proof of Thm:

$$L = \begin{bmatrix} R_1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & R_2 - a_{22} & & \\ & & \ddots & \\ -a_{n1} & & & R_n - a_{nn} \end{bmatrix} \quad \text{where } R_i := a_{i1} + a_{i2} + \cdots + a_{ii} + \cdots + a_{in} = \sum_{j=1}^{n} a_{ij}$$

$$= (R_{ii}\,\delta_{ij} - a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

$$\Rightarrow \det(\bar{L}^{n,n}) = \sum_{w \in \mathfrak{S}_{n-1}} \operatorname{sgn}(w) \prod_{i=1}^{n-1} L_{i,w(i)}$$

$$= \sum_{\substack{S \subseteq [n-1] \\ (\text{fixed by } w)}} \prod_{i \in S} (R_i - a_{ii}) \sum_{\substack{w \in \mathfrak{S}_{[n-1]\setminus S} \\ \text{a derangement}}} \operatorname{sgn}(w) \prod_{i \in [n-1]\setminus S} (-a_{i,w(i)})$$

$$= \sum_{S \subseteq [n-1]} \sum_{T \subseteq S} \prod_{i \in T} R_i \prod_{i \in S\setminus T} (-a_{ii}) \sum_{\substack{\text{derangement} \\ w \in \mathfrak{S}_{[n-1]\setminus S}}} \operatorname{sgn}(w) \prod_{i \in [n-1]\setminus S} (-a_{i,w(i)})$$

$$= \sum_{T \subseteq [n-1]} \underbrace{\prod_{i \in T} (a_{i1} + a_{i2} + \cdots + a_{in})}_{\|} \cdot \sum_{w \in \mathfrak{S}_{[n-1]\setminus T}} \operatorname{sgn}(w) \prod_{i \in [n-1]\setminus T} (-a_{i,w(i)})$$

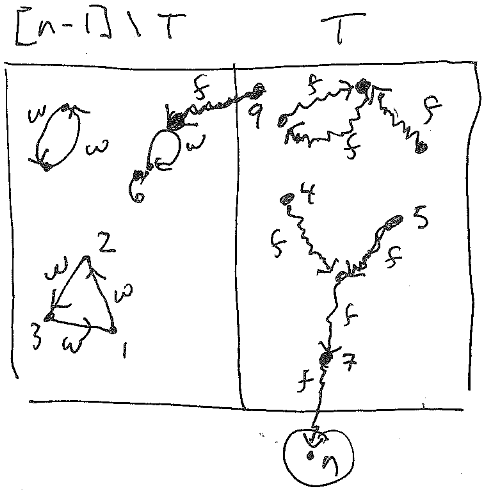
$$\sum_{f : T \to [n]} \prod_{i \in T} a_{i,f(i)}$$

$$= \sum_{(T,f,w)} \underbrace{(-1)^{|[n-1]\setminus T|} \operatorname{sgn}(w)}_{\operatorname{sgn}(x)} \underbrace{\prod_{i \in T} a_{i,f(i)} \prod_{i \in [n-1]\setminus T} a_{i,w(i)}}_{wt(x)}$$

$$X := \left\{ \begin{array}{l} (T,f,w) \\ T \subseteq [n-1] \\ f : T \to [n] \\ w \in \mathfrak{S}_{[n-1]\setminus T} \end{array} \right.$$

We will evaluate this signed, weighted sum using a sign-reversing involution...

Picture of (T, f, w):

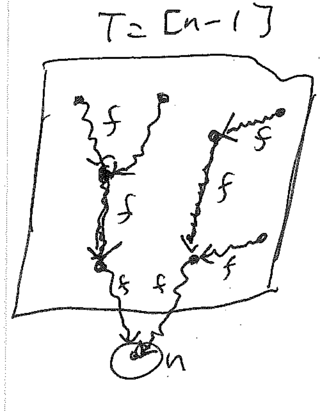


We can define an involution $\tau : X \to X$ that eliminates all cycles in w or f by switching them from w to f or back from f to w whichever cycle contains the smallest index $i \in [n-1]$.

Check that τ
- is an involution (clear)
- is wt-preserving (preserves arcs)
- is sign-reversing (sgn of a k-cycle is $(-1)^{k+1}$ ✓)

What are the fixed points X^{τ}?

No cycles if w or f $\Rightarrow$ $[n-1] \setminus T$ is empty, i.e. $T = [n-1]$ and $f : [n-1] \to [n]$ has no cycles.

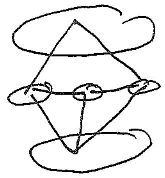


(easy) Lemma

This forces f to be an aborescence directed toward n (and conversely, any aboresence is such an f).

Hence $$\det(L^{n,n}) = \sum_{\substack{\text{arborescences} \\ f \text{ on } [n] \text{ directed} \\ \text{toward } n}} \prod_{i \in [n-1]} a_{i, f(i)}.$$

name comes from famous "Bridges of Königsberg" problem:



Digression on Euler tours + the BEST theorem (Ardil §3.1.4)

Kirchoff's thm (in its directed version) lets us solve another, seemingly unrelated problem:

Given a directed graph $D = (\overset{\text{vertices}}{V}, \overset{\text{arcs } (x,y) \; x \to y}{A})$, how many Euler tours (= circularly ordered walks along directed arcs in A visiting each arc exactly once, returning to start vertex) does it have?

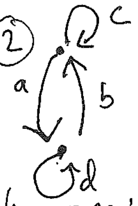
EXAMPLES:

①

has one:

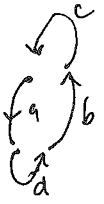
(abc)

②



has one:

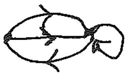
(adbc)
= (dbca)



④

has two: (adbec) ≠ (adbce)

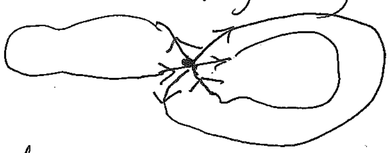
② but



has none.

Prop: D has an Euler tour ("D is Eulerian") $\Longleftrightarrow$
- its underlying undirected graph is connected, and
- $\text{outdeg}_D(v) = \text{indeg}_D(v) \quad \forall v \in V$.

Proof: ($\Rightarrow$) is pretty clear, since the tour connects V and matches outgoing w/ incoming arcs at each $v \in V$ like this:



($\Leftarrow$) If outdeg = indeg everywhere, pick v_0 to start and leave along any arc (then erase it), entering v_1, and then leaving along some arc (then erase it). Repeat until you get stuck, which can only be at v_0 since outdeg = indeg is preserved. This creates a directed cycle $\mathcal{C}$, and D being connected means that either C exhausts D, or some vertex on C has an arc not in C. Start walking there (w/ C erased) to produce a new cycle C'. Then "suture" the cycles C and C' like this:

$\rightsquigarrow$

(or just "concatenate" the cycles)

Repeat until D is exhausted. ▨

11/6 Thm (B.E.S.T.) (de Bruijn, van Aardenne-Ehrenfest, Smith, Tutte)

If D has an Euler tour, then it has

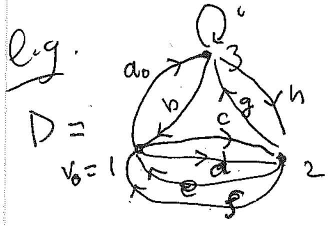
$$\underbrace{\#\left(\begin{smallmatrix}\text{aboresecences in } D\\ \text{directed toward fixed } v_0\end{smallmatrix}\right)}_{\text{easy to compute (Kirchoff)}} \cdot \underbrace{\prod_{v\in V}(\text{outdeg}_D(v)-1)!}_{\text{even easier!}} \text{ of them.}$$

Proof: Start all tours at some fixed arc a_0 emanating from v_0 (by convention).

Given an Euler tour t in D, create

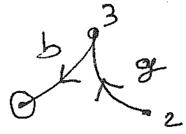
- $\alpha(t) := \{$ the set of one arc for each $v \neq v_0$ which is the last arc out of $v \underset{v}{\longrightarrow}$ visited by t $\}$
- $(W_v(t))_{v\in V} := \{$ the linear order on the non-$\alpha(t)$ non-a_0 arcs out of v in the order in which t visits them $\}$

e.g. D =



w/ t =

has α(t) =



$= (a_0 i h f d e c g b)$ and

$W_{v_0}(t) = (d, c)$ (omitting a_0)

$W_2(t) = (f, e)$ (omitting g)

$W_3(t) = (i, h)$ (omitting b)

Claim: α(t) is always an aboresence in D directed towards v_0, since it has exactly $|V|-1$ arcs (one for each $v \in V - \{v_0\}$), and has a path $v \to \cdots \to v_0$ for every $v \in V$ (by backwards induction on how late v is visited by t)

Thus we get a map
$$\left\{\begin{matrix}\text{Euler tours}\\ \varepsilon \text{ in } D\end{matrix}\right\} \xrightarrow{f} \left\{ (\alpha, (w_v)_{v\in V}) : \begin{matrix}\alpha \text{ arborescence in } D \text{ directed towards } v_0,\\ \text{and } (w_v)_{v\in V} \text{ linear order for each } v\in V\\ \text{of the non-}\alpha \text{ arcs leaving } v\end{matrix} \right\}$$
non-αs

__Claim__ f is invertible, i.e. __every__ $(\alpha, (w_v))$ determines a unique ε.

(Pf by example here... let the "audience" pick $(\alpha, (w_v))$ and compute $\varepsilon = f^{-1}((\alpha, (w_v)))$)

This finishes the pf, since image of f has desired cardinality. ▨

[__N.B.__: Computing # Euler tours of an __undirected__ graph is #P-complete!]
big __contrast__

④ __Lindström - Gessel - Viennot Lemma__:

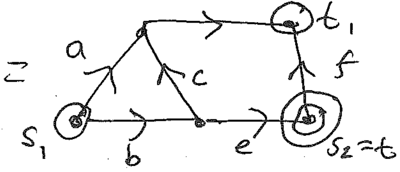
mma

Let D be an __acyclic__ digraph with distinguished vertices $\overset{\text{sources}}{s_1, \ldots, s_n}$, $\underset{\hookrightarrow \text{sinks}}{t_1, \ldots, t_n}$

If $M = (m_{ij})_{\substack{i=1,\ldots,n\\ j=1,\ldots,n}}$ has $m_{ij} := \sum_{\substack{\text{paths } P \text{ in}\\ D \text{ from } s_i \text{ to } t_j}} \underbrace{w(P)}_{:= \prod_{\text{arcs } a \text{ in } P} a}$

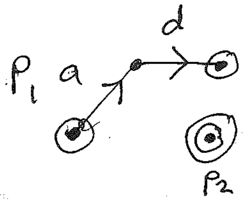
then $\det M = \sum_{\substack{\text{vertex-disjoint}\\ \text{paths } (P_1, \ldots, P_n):\\ P_i : s_i \to t_{w(i)}}} \operatorname{sgn}(w) \prod_{i=1}^{n} w(P_i)$.

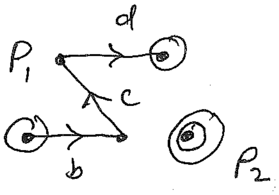
e.g. $D =$



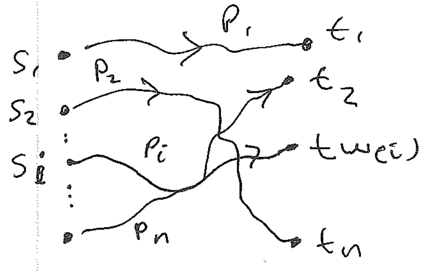
$\rightsquigarrow M = \begin{array}{c|cc} & t_1 & t_2 \\ \hline s_1 & ad+bcd+bef & be \\ s_2 & f & 1 \end{array}$

has $\det(M) = (ad + bcd + b\cancel{ef}) \cdot 1 - \cancel{bef}$
$= ad + bcd$





$$\underline{\text{Pf:}}\quad \det M = \sum_{w \in \mathfrak{S}_n} \operatorname{sgn}(w) \prod_{i=1}^{n} \underbrace{M_{i,w(i)}}_{\sum_{P: s_i \to t_{w(i)}} w(P)} = \sum_{X := \left\{ \substack{\text{paths } (P_1,\dots,P_n) \\ P_i : s_i \to t_{w(i)}} \right\}} \operatorname{sgn}(w) \prod_{i=1}^{n} w(P_i)$$

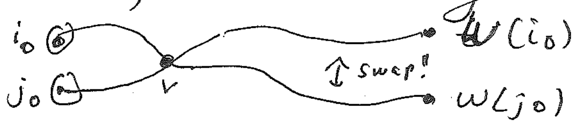


Want to define an involution $\tau : X \to X$ cancelling down to $X^{\tau} = \{\text{vertex-disjoint } (P_1,\dots,P_n)\}$

If $(P_1, \dots, P_n)$ are not vertex disjoint:

- find P_{i_0} w/ smallest i_0 intersecting another path,
- find earliest $v \in P_{i_0}$ that's an intersection point,
- find P_{j_0} w/ smallest $j_0 \neq i_0$ s.t. $v \in P_{j_0}$,

an then keep all other paths the same, while having P_{i_0} and P_{j_0} exchange the tails of their paths after the intersection v:

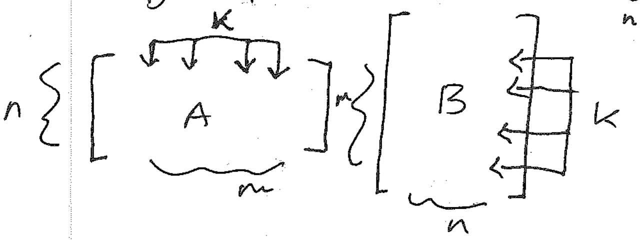


▨

11/8

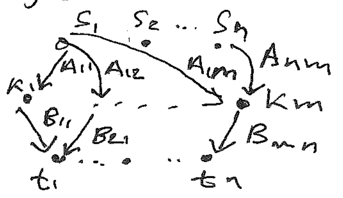
Cor 1 (Cauchy-Binet Thm)

If A $n \times m$, B $m \times n$ then $\det(\underbrace{AB}_{n \times n}) = \sum_{\substack{K \subseteq [m] \\ |K| = n}} \det(A|_{\text{cols } K}) \det(B|_{\text{rows } K})$



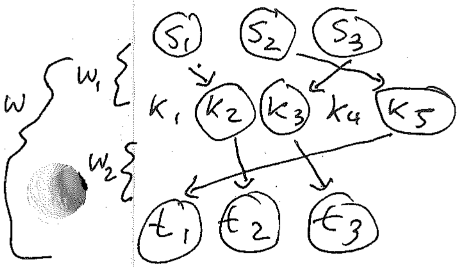
$$\underline{\text{Pf:}}\ (AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{jk}$$

$$= \sum_{\substack{\text{paths } P: s_i \to t_j \\ \text{in } D}} w(P) \quad \text{w/ } D =$$



and hence $\det(AB) = \sum_{\substack{\text{vertex-disjoint} \\ (P_1,\dots,P_n) \\ P_i : s_i \to t_{w(i)}}} \operatorname{sgn}(w) \prod_{i=1}^{n} w(P_i)$

$$= \sum_{\substack{K \subseteq [m] \\ \shortparallel \\ \{k_1 < k_2 < \cdots < k_n\}}} \underbrace{\left(\sum_{\substack{w_1 \in \mathfrak{S}_n \\ \shortparallel \\ \text{bij.} [n] \to K \\ \{s_1,\dots,s_n\}}} \operatorname{sgn}(w_1) \prod_{i=1}^{n} A_{i,w_1(i)} \right)}_{\det(A|_{\text{cols } K})} \underbrace{\left(\sum_{\substack{w_2 \in \mathfrak{S}_n \\ = \text{bij.} \\ K \to [n] \\ \{t_1,\dots,t_n\}}} \operatorname{sgn}(w_2) \prod_{i=1}^{n} B_{i,w_2(i)} \right)}_{\det(B|_{\text{rows } K})}$$

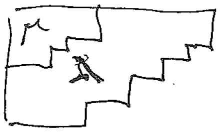


$K = \{2, 3, 5\}$

▨

~~Cor~~ 2 (Jacobi-Trudi formula)

Given partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ w/ $\mu_i \leq \lambda_i \ \forall i$
$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell)$,



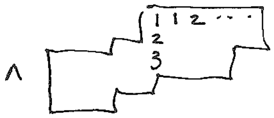
then defining $h_r(x_1,\ldots,x_n)$:= complete homogeneous symmetric polynomial of deg r
(for $r \geq 1$)
$$= \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} = x_1^r + x_1^{r-1} x_2 + \cdots + x_1 x_2 \cdots x_r + \cdots + x_n^r$$

and $h_0(x_1,\ldots,x_n) := 1$, and $h_{-r}(x_1,\ldots,x_n) := 0$

then $$\det\left(h_{(\lambda_i - i)-(\mu_j - j)}(x_1,\ldots,x_n)\right)_{\substack{i=1,\ldots,\ell \\ j=1,\ldots,\ell}} = \sum_{\substack{\text{column-strict} \\ \text{tableaux } T \\ \text{of shape } = \lambda/\mu \\ \text{w/ entries in } [n]}} \prod_{i \in T} x_i \left(= \frac{\text{skew Schur polynomial}}{s_{\lambda/\mu}(x_1,\ldots,x_n)} \right)$$

(also called "semistandard" tableaux)

$\rightsquigarrow$



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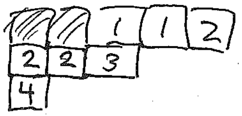
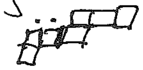
e.g. $\lambda = (5,3,1)$, $\mu = (2,0,0)$, $n = 4$

one $T =$



$\rightsquigarrow \prod_{i \in T} x_i = x_1^2 x_2^3 x_3 x_4$,

s



$(x_1,\ldots,x_4) =$

$$\det\begin{bmatrix} h_3 & h_6 & h_7 \\ 1 & h_3 & h_4 \\ 0 & 1 & h_1 \end{bmatrix}$$

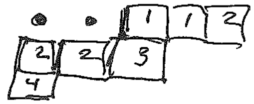
$\|$

Lemma: $$\det\begin{bmatrix} h_{5-2}^{(x_1,\ldots,x_4)} & h_{5-0+1} & h_{5-0+2} \\ h_{3-2-1} & h_{3-0} & h_{3-0+1} \\ h_{1-2-2} & h_{1-0-1} & h_{1-0} \end{bmatrix}$$

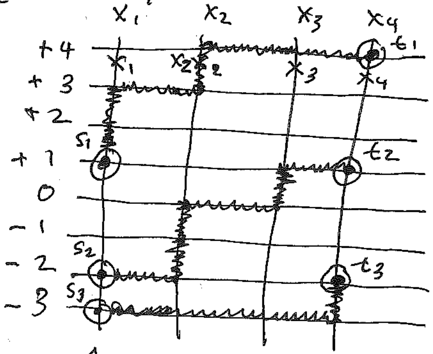
Pf: $\mu = (2,0,0) \rightsquigarrow \overset{s_i\text{'s}}{(+1,-2,-3)}$

$\lambda = (5,3,1) \rightsquigarrow (+4,+1,-2) \leftarrow t_i$'s

col.-strict tableau $T =$



$\longleftrightarrow$ exercise



vertex disjoint paths $(P_1,\ldots,P_\ell)$ where P_i's vertical steps are dictated by row i

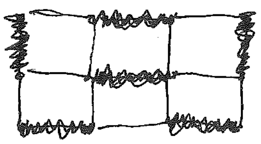
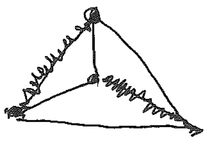
Let D be rectangular grid w/ arrows ↑ and →, having variables $x_1,\ldots,x_n$ on the ↑ arrows, and 1 on the → arrows, w/ $(s_1,\ldots,s_\ell)$ on the x_1-vertical at heights $\mu - (1,2,\ldots,\ell)$ and $(t_1,\ldots,t_\ell)$ on the x_n-vertical at heights $\lambda - (1,2,\ldots,\ell)$.

Then note $$h_{\underbrace{(\lambda_i - i)}_{\substack{\text{height} \\ t_i}} - \underbrace{(\mu_j - j)}_{\substack{\text{height} \\ \text{of } s_j}}}(x_1,\ldots,x_n) = \sum_{\substack{\text{Path } P: \\ s_j \to t_i}} wt(P)$$ + apply LGV. ▨

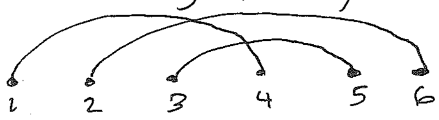
11/11

⑤ Pfaffians and matchings (Ardila § 3.1.5)

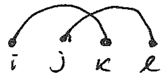
DEF'n In a graph $G=(V,E)$ a (perfect) matching $M \subseteq E$ is a set of edges for which $\deg_M(v)=1$ $\forall v \in V$.

e.g. $G =$  or $G =$  (edges in M = 〰)

A matching M in $K_{2n} = ([2n], \{\text{all pairs } \{i,j\}\})$ will be depicted by putting V on a line, w/ arcs  in upper half-plane:

e.g.  $= M$

Its crossing number $cr(M) :=$ # crossings of arcs (drawn generically)

$= \# \{$  $i<j<k<\ell : \{i,k\}, \{j,\ell\} \in M \}$

Prop The generic skew symmetric matrix $A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1N} \\ -a_{12} & 0 & a_{23} & \cdots & \\ -a_{13} & & & & \\ \vdots & & & & \\ -a_{1N} & & \cdots & & 0 \end{bmatrix} (=-A^T)$

has $\det(A) = 0$ if N is odd

$\det(A) = Pf(A)^2$ if $N = 2n$ is even,

Pfaffian

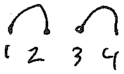
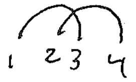
where $Pf(A) := \sum_{\substack{\text{matchings} \\ M \subseteq K_{2n}}} (-1)^{cr(M)} \prod_{\{i,j\} \in M} a_{i,j}$.

e.g. $N=2$ $\det \begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix} = a_{12}^2$ $Pf(A) = a_{12}$

 M

$N=3$ $\det \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = -a_{12}a_{23}a_{13} + a_{12}a_{23}a_{13} = 0$

$N=4$ $\det \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} = (\overbrace{a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}}^{Pf(A)})^2$

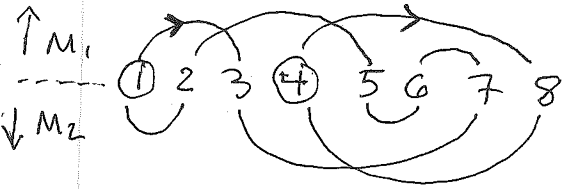
<u>Proof</u>: If N is odd, then $\det(A) = \det(A^t) = \det(-A) = (-1)^N \det(A) = -\det(A)$
$\Rightarrow \det(A) = 0$

For $N = 2n$ even, want $\det(A) = \sum_{w \in \mathfrak{S}_{2n}} \mathrm{sgn}(w) \prod_{i=1}^{2n} a_{i,w(i)} \overset{?}{=} \mathrm{Pf}(A)^2$

with the convention $a_{j,i} = -a_{i,j}$ if $i<j$ (so $a_{i,i} = 0$)

$$= \sum_{\substack{\text{matchings} \\ (M_1,M_2) \text{ in } K_{2n}}} (-1)^{cr(M_1)+cr(M_2)} \prod_{\substack{\{i,j\} \in M_1 \sqcup M_2 \\ 1 \le i<j \le 2n}} a_{i,j}$$

A pair (M_1, M_2) of matchings gives rise to a $w \in \mathfrak{S}_{2n}$ by orienting the cycles in $M_1 \sqcup M_2$:



$\rightsquigarrow w = (\textcircled{1} \overset{M_1}{\frown} 3\ 7\ 6\ 5\ 2)(\textcircled{4} \overset{M_1}{\frown} 8)$

<u>Claim</u>: $(-1)^{cr(M_1)+cr(M_2)} \prod_{\{i,j\} \in M_1 \sqcup M_2} a_{ij} = \mathrm{sgn}(w) \prod_{i=1}^{2n} a_{i,w(i)}$

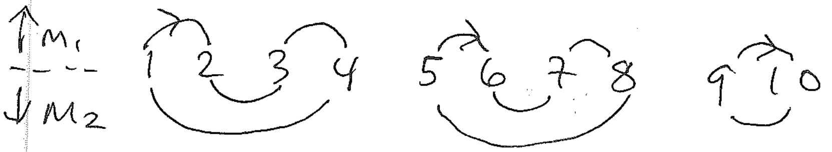
e.g. $(-1)^{2+1} a_{13} a_{37} a_{67} a_{56} a_{25} a_{48}^2 = (-1)^2 a_{13} a_{37} \underset{}{\overset{-a_{67}}{\widetilde{a_{76}}}} \overset{-a_{56}}{\widetilde{a_{65}}} \overset{-a_{25}}{\widetilde{a_{52}}} \overset{-a_{12}}{\widetilde{a_{21}}} a_{48} \overset{-a_{48}}{\widetilde{a_{84}}}$

Note that <u>Claim</u> is equivalent to: $(-1)^{cr(M_1)+cr(M_2)} \overset{?}{=} \mathrm{sgn}(w) \cdot (-1)^{\text{non-exc}(w)}$
which one can prove by noting that:

(i) LandRHS change by ± 1 (same sign) if one conjugates w by an adjacent transposition $(i, i+1)$
(namely, change by $\begin{cases} -1 & \text{if } (i,i+1) \text{ matched in } M_1 \text{ XOR } M_2 \\ +1 & \text{otherwise} \end{cases}$)

(ii) so by conjugating, one can make w some canonical permutation of given cycle type λ, and check for this w:

e.g. $\lambda = (4, 4, 2)$



$\rightsquigarrow w = (1234)(5678)(9\,10)$

$(-1)^{cr(M_1)+cr(M_2)} \overset{?}{=} \mathrm{sgn}(w) \cdot (-1)^{\text{non-exc}(w)}$

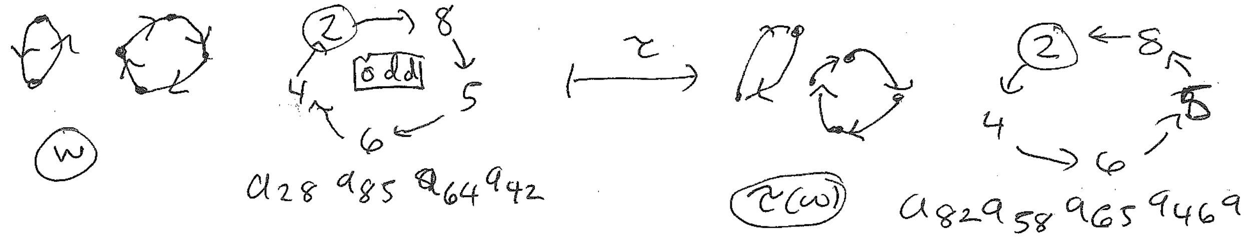
$(-1)^0 = (-1)^{1+1} \cdot (-1)^{1+1} \cdot (-1)^{1+1}$

$\| \qquad = 1$ ✓

1

Now, need only define a sign-reversing involution $\tau: \overset{G_{2n}}{\overset{\shortparallel}{X}} \to X$ that cancels w having at least one odd cycle (the "only even cycles" permutations exactly correspond to pairs (M_1, M_2)):

- to define τ, find the odd cycle in w w/ the smallest entry, and reverse its arrows:



$a_{28} a_{85} a_{64} a_{42}$

$a_{82} a_{58} a_{65} a_{46} a_{24}$
$= (-1)^5 a_{28} a_{85} a_{56} a_{64} a_{42}$

note $a_{ii} = 0$ for possible ~~fixed-pts~~

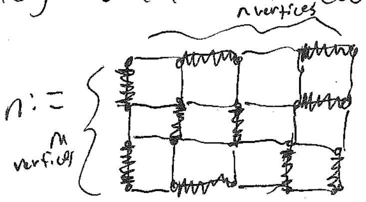
Can check that this τ indeed reverses sign + preserves weight. ▨

11/13

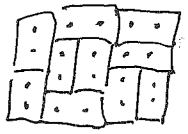
Cultural digression: Kasteleyn's method for the dimer problem (the "permanent-determinant"/"Pfaffian-Hafnian" method)

Kasteleyn wanted to count the number of perfect matchings in

$G_{m,n} :=$



(biject w/ domino tilings of rectangle →



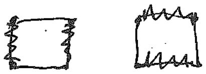
)

and other graphs G; called the dimer problem for G.

WLOG m is even (else $|V| = mn$ odd if both m, n odd).

e.g. we saw earlier w/ $m=2$ one gets Fibonacci #'s

$n=2$:



2, $n=3$:

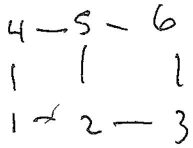
3

His idea was to start w/ the skew-symmetric matrix

$$(A_G)_{i,j} := \begin{cases} a_{ij} = -a_{ji} & \text{if } i<j \text{ and } \{i,j\} \in G \\ 0 & \text{if } \{i,j\} \notin G, \end{cases}$$

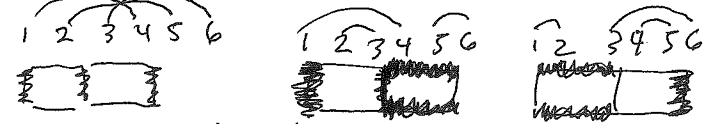
and its Pfaffian, which counts matchings w/ unwanted signs.

e.g. $G =$



$\rightsquigarrow A_G = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & a_{12} & & a_{14} & & \\ 2 & -a_{12} & 0 & a_{23} & & a_{25} & \\ 3 & & -a_{23} & 0 & & & a_{36} \\ 4 & -a_{14} & & & 0 & a_{43} & \\ 5 & & -a_{25} & & -a_{45} & 0 & a_{56} \\ 6 & & & -a_{36} & & -a_{56} & 0 \end{array}$

has $Pf(A_G) \left(= \pm\sqrt{\det(A_G)}\right) = -a_{14}a_{25}a_{36} + a_{14}a_{23}a_{56} + a_{12}a_{36}a_{45}$

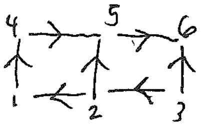


But it would be ~~fixed~~ fixed if all terms had same sign,
e.g. $+a_{12} \to -a_{12}$, $+a_{23} \to -a_{23}$.

DEF'n Given $G=(V,E)$ undirected, and $D=(V,A)$ directing E (an orientation)
Create S_D, skew-symmetric matrix

$$(S_D)_{i,j} := \begin{cases} +a_{ij} & \text{if } i<j \text{ and } i \to j \text{ in } D \\ -a_{ij} & \text{if } i<j \text{ and } i \leftarrow j \text{ in } D \\ 0 & \text{if } \{i,j\} \notin E. \end{cases}$$

e.g. $D =$



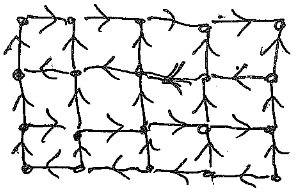
$\rightsquigarrow S_D = \begin{bmatrix} 0 & -a_{12} & & a_{14} & & \\ a_{12} & 0 & -a_{23} & & a_{25} & \\ & a_{23} & 0 & & & a_{36} \\ -a_{14} & & & 0 & a_{45} & \\ & -a_{23} & & -a_{45} & 0 & a_{56} \\ & & -a_{36} & & -a_{56} & 0 \end{bmatrix}$

and $Pf(S_D) = -(\underbrace{a_{14}a_{23}a_{36} + a_{14}a_{23}a_{56} + a_{12}a_{36}a_{45}}_{\text{all same sign}})$

DEF'n Say D is a Pfaffian orientation of G if all terms of $Pf(S_D)$ have same signs.

Thm (Kasteleyn) Every planar graph G has a Pfaffian orientation.
(See Loehr "Bijective combinatorics" for pf.)

e.g. for $G_{m,n} =$



- up in columns
- alternate right/left in rows

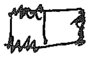
turns out to work (not obvious!)

Remark: This shows that one can count perfect matchings for planar graphs in polynomial time, by computing $|Pf(S_D)_{a_{ij}=1}| = \left|\sqrt{\det(S_D)_{a_{ij}=1}}\right|$. By contrast, counting matchings of arbitrary G is a #P-complete problem!

Thm (Kasteleyn)

$$\sum_{\substack{\text{matchings } M \\ \text{in } G_{m,n}}} x^{\#\text{vertical edges in } M}\, y^{\#\text{horiz. edges in } M} = 2^{\frac{mn}{2}} \prod_{j=1}^{m/2} \prod_{k=1}^{n} \sqrt{x^2\cos^2\left(\frac{j\pi}{m+1}\right) + y^2\cos^2\left(\frac{k\pi}{n+1}\right)}$$

e.g. for $m=2, n=3$

LHS $= \Big\{$  x^3  $+ xy^2$  $+ xy^2 \Big\} = x^3 + 2xy^2$

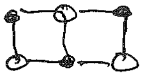
$$\text{RHS} = 2^{\frac{2\cdot 3}{2}} \prod_{j=1}^{2/2} \prod_{k=1}^{3} \sqrt{x^2\cos^2\left(\frac{j\pi}{3}\right) + y^2\cos^2\left(\frac{k\pi}{4}\right)}$$
$$= 8 \prod_{k=1}^{3} \sqrt{x^2\left(\tfrac{1}{4}\right) + y^2\cos^2\left(\tfrac{k\pi}{4}\right)}$$
$$= 8\sqrt{\tfrac{x^2}{4}+\tfrac{y^2}{2}} \cdot \sqrt{\tfrac{x^2}{4}+0}\,\sqrt{\tfrac{x^2}{4}+\tfrac{y^2}{2}} = 8\left(\tfrac{x^2}{4}+\tfrac{y^2}{2}\right)\left(\tfrac{x}{2}\right) = x^3+2xy^2 \checkmark$$

Pf idea: Compute eigenvalues/eigenvecters for relevant matrix S_D explicitly, and use the Pfaffian. ▨

Cor # matchings in $G_{m,n} \sim c \cdot e^{\frac{G}{\pi} mn}$, where $G = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots$ ← "Catalan's constant"

Pf: Take logs of product to convert to sum, estimate via an integral. ▨

Remarks ① If $A = \begin{array}{c} \begin{array}{cc} x_1\cdots x_n & y_1\cdots y_n \end{array} \\ \begin{bmatrix} 0 & B \\ -B^t & 0 \end{bmatrix} \end{array}$, then $\mathrm{Pf}(A) = \det(B)$. ← easy via matchings def'n of Pfaffian

② When G is bipartite (as is the case w/ $G_{m,n} =$ )

then one can write $S_D = \begin{array}{c} \begin{array}{cc} \text{black} & \text{white} \end{array} \\ \begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix} \end{array}$ so that $\mathrm{Pf}(S_D) = \det(A)$.

③ Why "Permanent - determinant" / "Pfaffian - Hafnian" method?

Recall $\underset{\text{permanent}}{\mathrm{Per}}(M) := \sum_{w \in \mathfrak{S}_n} + \prod_{i=1}^{n} m_{i,w(i)}$ ← same as determinant, but w/ all + signs

Similarly, $\underset{\text{Hafnian}}{\mathrm{Haf}}(A) = \sum_{\substack{\text{matchings} \\ M \text{ of } [2n]}} + \prod_{\substack{\{i,j\}\in M \\ i<j}} a_{ij}$

Kasteleyn's method evaluates a Hafnian as a Pfaffian of another matrix: $\mathrm{Haf}(A) = \mathrm{Pf}(A')$, or $\mathrm{Per}(M) = \mathrm{Det}(M')$ in bipartite case. //

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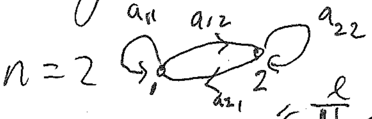
The transfer-matrix method (Stanley §4.7, Ardila §3.1.2)
another tool from linear algebra for counting walks in digraphs
(and other problems...)

Thm Given an $n \times n$ matrix $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$, think of a_{ij} as labeling arcs $\overset{i}{\circ} \to \overset{j}{\circ}$ in complete digraph (with loops) on $[n]$:

e.g. $n=2$



Then we have the following:

(a) $$\sum_{\substack{\text{directed walks of length } \ell \\ i =: i_0 \to i_1 \to \cdots \to i_{\ell-1} \to i_\ell =: j}} \overbrace{wt(P)}^{= \prod_{k=1}^{\ell} a_{i_{k-1}, i_k}} = (A^\ell)_{i,j} \text{ for all } \ell \geq 0$$

(b) $$\sum_{\text{walks } P \text{ from } i \text{ to } j} t^{\text{length}(P)} wt(P) = \frac{(-1)^{i+j} \det\left((I_n - tA) \text{ w/ } i\text{th row and } j\text{th column removed}\right)}{\det(I_n - tA)}$$

(c) $$\sum_{\text{closed walks } P} t^{\text{length}(P)} wt(P) = \sum_{\ell \geq 0} t^\ell (\lambda_1^\ell + \cdots + \lambda_n^\ell) = \frac{-t \, d/dt \det(I_n - tA)}{\det(I_n - tA)}$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of A

Proof: (a) is just definition of matrix multiplication:
$$(A^\ell)_{i,j} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{\ell-1}}^{n} a_{i,i_1} a_{i_1,i_2} \cdots a_{i_{\ell-2}, i_{\ell-1}} a_{i_{\ell-1}, j} = \text{LHS of (a)}$$ ✓

For (b), $$\text{LHS} \overset{\text{by (a)}}{=} \sum_{\ell \geq 0} t^\ell (A^\ell)_{ij} = \left(\sum_{\ell \geq 0} t^\ell A^\ell\right)_{i,j}$$

$$= (I_n + tA + t^2A^2 + \cdots)_{i,j} = \left[(I_n - tA)^{-1}\right]_{ij}$$

$$\boxed{\begin{array}{c}\text{by Cramer's Rule:} \\ \text{adj } B \cdot B = \det B \cdot I_n \\ (\text{adjugate matrix of } B)\end{array}} \longrightarrow = \frac{(-1)^{i+j} \det\left((I_n - tA) \text{ w/ } \begin{smallmatrix} i\text{th row} \\ j\text{th col}\end{smallmatrix} \text{ removed}\right)}{\det(I_n - tA)}$$

For (c), $$\text{LHS} \overset{\text{by (a)}}{=} \sum_{i=1}^{n} \sum_{\ell \geq 1} t^\ell (A^\ell)_{i,i} = \sum_{\ell \geq 1} t^\ell \operatorname{trace}(A^\ell)$$

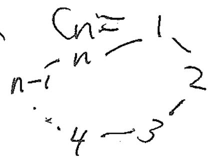
$$\boxed{\begin{array}{l}\text{since if } PAP^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \\ \text{then } PA^\ell P^{-1} = \begin{bmatrix} \lambda_1^\ell & & * \\ & \ddots & \\ & & \lambda_n^\ell \end{bmatrix}\end{array}} \longrightarrow = \sum_{\ell \geq 1} t^\ell (\lambda_1^\ell + \cdots + \lambda_n^\ell) = \frac{\lambda_1 t}{1 - \lambda_1 t} + \cdots + \frac{\lambda_n t}{1 - \lambda_n t}$$

$$= \frac{t \sum_{k=1}^{n} \lambda_k (1 - \lambda_1 t) \cdots \widehat{(1 - \lambda_k t)} \cdots (1 - \lambda_n t)}{\prod (1 - \lambda_k t)}$$

$$= \frac{-t \, d/dt \prod_{k=1}^{n} (1 - \lambda_k t)}{\prod_{k=1} (1 - \lambda_k t)} = \frac{-t \, d/dt \det(I_n - tA)}{\det(I_n - tA)}.$$ ☒

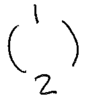
EXAMPLE: Chromatic polynomial of cycle graph.

Let $f(n,k) :=$ # of proper vertex-colorings of $C_n =$ w/ k colors



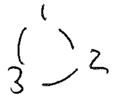
↗ no adjacent vertices w/ same color

e.g. $n=2$



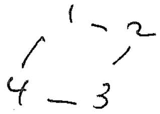
$$f(2,k) = \underset{\substack{\text{color 1}\\ \text{first in } k \text{ ways}}}{k}\underset{\substack{\text{color 2}\\ \text{differently}}}{(k-1)} = (k-1)k$$

$n=3$



$$f(3,k) = \underset{\text{color 1}}{k}\ \underset{\text{color 2}}{(k-1)}\ \underset{\text{color 3}}{(k-2)} = (k-1)(k^2-2k)$$

$n=4$



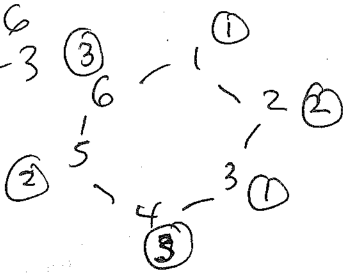
$$f(4,k) = \underbrace{\overset{\text{color 1}}{k}\overset{\text{color 2}}{(k-1)}\overset{\text{color 4}}{(k-2)}\overset{\text{color 3}}{(k-2)}}_{\substack{2+4 \text{ have}\\ \text{different colors}}} + \overset{\text{color 1}}{k}\underbrace{\overset{\text{color 2+4 same}}{(k-1)}\overset{\text{color 3}}{(k-1)}}_{\substack{2+4 \text{ have}\\ \text{same color}}}$$

$$= k(k-1)\left((k-2)^2+k-1\right) = (k-1)\left(k^3-3k^2+3k\right)$$

<u>Note</u>: $\{$proper k-colorings of $C_n\} \longleftrightarrow \{$closed walks of length n in $\overleftrightarrow{K_k} =$ complete directed graph w/ no loops$\}$

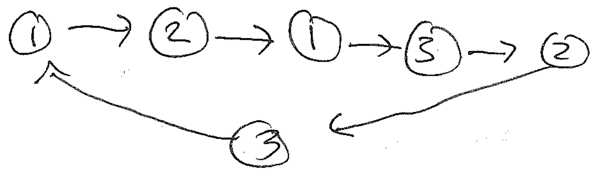
if the coloring assigns vertex $i \in [n]$ to color $j \in [k]$, then the walk visits vertex j of $\overleftrightarrow{K_k}$ at its i^{th} step:

e.g. $n=6$, $k=3$

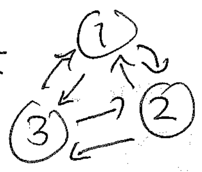


↗ proper 3-coloring of C_6

$\longleftrightarrow$



a closed walk in $\overleftrightarrow{K_3} =$



of length 6

So taking $A = \begin{array}{c|cccc} & ① & ② & \cdots & ⓚ \\ \hline ① & 0 & 1 & \cdots & 1 \\ ② & 1 & 0 & 1 \cdots & 1 \\ \vdots & \vdots & 1 & 0 \ddots & \vdots \\ ⓚ & 1 & 1 & \cdots 1 & 0 \end{array} = \mathbb{1}_k - I_k,$

which has eigenvalues $(\lambda_1, \dots, \lambda_k) = (k-1, \overbrace{-1, -1, \dots, -1}^{k-1 \text{ terms}})$
(since we already saw $\mathbb{1}_k$ has eigen's $(k, \overbrace{0, \dots, 0}^{k-1})$)

One finds that $f(n,k) = \lambda_1^n + \cdots + \lambda_k^n$

$$= (k-1)^n + (-1)^n + \cdots + (-1)^n$$
$$= (k-1)^n + (k-1)(-1)^n$$
$$= (k-1)\left((k-1)^{n-1} + (-1)^n\right)$$

e.g. $n=2$ $f(2,k) = (k-1)(k-1+1) = (k-1)(k)$

$n=3$ $f(3,k) = (k-1)((k-1)^2-1) = (k-1)(k^2-2k)$

$n=4$ $f(4,k) = (k-1)((k-1)^3+1) = (k-1)(k^3-3k^3+3k)$ ✓

Remark: We saw that k-colorings of C_n are the same ~~[illegible]~~ length n words $w = (w_1, \dots, w_n)$ in alphabet $\{1, 2, \dots, k\}$ s.t. $w_i \neq w_{i+1}$ ~~for~~ all $i = 1, \dots, n-1$ and $w_n \neq w_1$.

Collection of such words^{t_n} is an example of a regular language (notion from theoretical computer science).

Other regular languages:
- words in $\{0,1\}$ avoiding 00 and 1010 as consecutive substring
- words in $\{0,1\}$ w/ an even # of 0's

etc... ("finite amount of memory")

Transfer-matrix method applies to all regular languages, and in particular shows they have rational (ordinary) generating functions.

[Other levels of "Chomsky hierarchy" are also interesting from enumerative point of view
- regular language $\leftrightarrow$ finite-state automaton $\leftrightarrow$ rational g.f.
- context-free grammar $\leftrightarrow$ push-down automaton $\leftrightarrow$ algebraic g.f. (if unambiguous)

decidable language $\leftrightarrow$ Turing machine $\leftrightarrow$??]

$\prod_{k=1} (1-\lambda_k t)$