11/18

New + final topic for the class: **Posets** (Stanley Ch. 3, Ardila §4)

not nec. finite!

DEF'N Recall a poset $(P, \leq)$ is a binary relation $x \leq y$ on a set P

which is
- reflexive $x \leq x$
- antisymmetric $x \leq y,\ y \leq x \Rightarrow x = y$
- transitive $x \leq y,\ y \leq z \Rightarrow x \leq z$

Examples

① $([n] := \{1, 2, \ldots, n\}, \leq_{[n]})$

$\cap$

$(\mathbb{N}, \leq_{\mathbb{N}})$

$\cap$

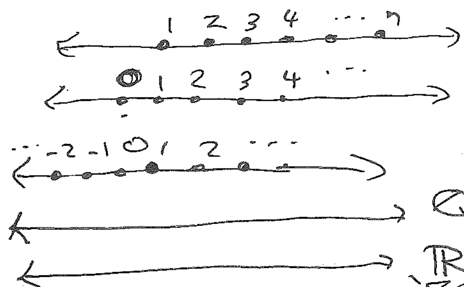
$(\mathbb{Z}, \leq_{\mathbb{Z}})$

$\cap$

$(\mathbb{Q}, \leq_{\mathbb{Q}})$

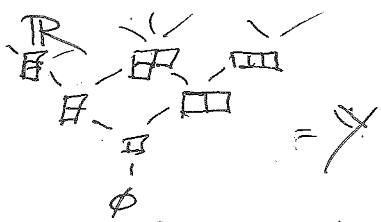
$\cap$

$(\mathbb{R}, \leq_{\mathbb{R}})$



totally / linearly ordered set; "a chain"

② $\mathbb{Y}$:= Young's lattice of all partitions λ



③ For S a set, $(2^S, \leq)$ = Boolean algebra on {all subsets of S} w/ $S \leq T$ if $S \subseteq T$

When $S = [n]$, we will write $2^S =: B_n$ ("n^{th} finite Boolean algebra")

e.g. $B_1 = $ {1} over $\varnothing$, $B_2 = $ {1,2} over {1}, {2} over $\varnothing$, etc...

Some common poset properties:

	acc = ascending chain condition (no ∞ chains $x_1 \lneq x_2 \lneq \cdots$)	dcc = descending chain condition (no ∞ chains $x_1 \gneq x_2 \gneq \cdots$)	chain-finite = acc + dcc	locally finite = all intervals $[x,y] := \{z : x \leq z \leq y\}$ are finite	∃ bottom element $\hat{0}$	∃ top element $\hat{1}$
$[n]$	yes	yes	yes	yes	yes, $\hat{0} = 1$	yes, $\hat{1} = n$
$\mathbb{N}$	no	yes	no	yes	yes, $\hat{0} = 0$	no
$\mathbb{Z}$	no	no	no	yes	no	no
$\mathbb{Q}, \mathbb{R}$	no	no	no	no	no	no
B_n	yes	yes	yes	yes	yes, $\hat{0} = \varnothing$	yes, $\hat{1} = [n]$
2^S, $\lvert S \rvert = \infty$	no	no	no	no	yes, $\hat{0} = \varnothing$	yes, $\hat{1} =$
$\mathbb{Y}$	no	yes	no	yes	yes, $\hat{0} = \varnothing$	no

When P is locally-finite (or even just locally chain-finite, i.e., all intervals $[x,y]$ are chain finite), then $\leq_P$ is the transitive closure of the covering relation $x \lessdot_P y$ defined by

- $x \lessdot_P y \iff x \lneq_P y$ and $\nexists\, z$ in P w/ $x \lneq_P z \lneq_P y$.

Then, as we have seen, one can represent P by its Hasse diagram:

draw P as nodes in the plane w/ edges x — y whenever $x \lessdot_P y$, (and draw y higher in the plane)

DEF'N Say P is graded if we can write $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ for some n, or $P = P_0 \sqcup P_1 \sqcup \cdots$, so that every maximal chain (= totally ordered subset) in P has form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$, $x_i \in P_i$ or $x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots$, $x_i \in P_i$.

In this case, $\exists$ unique rank function $\rho : P \to \{0,1,2,\ldots\}$ satisfying $\rho(x) = 0$ iff x is minimal in P and $\rho(y) = \rho(x)+1$ if $y \gtrdot_P x$.

(Namely, set $\rho(x) = i$ if $x \in P_i$.)

For P graded, define rank generating-fn. $F(P,x) := \sum_{p \in P} x^{\rho(p)}$ (also denoted "rank(p)").

Examples We've seen several examples already...

① $F(B_n, x) = \sum_{k=0}^{n} \binom{n}{k} x^n$

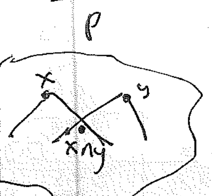
② $F(\mathbb{Y}, x) = \sum_{n=0}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)}$ (partition number)

③ $F(\mathfrak{S}_n, \leq_{abs}, x) = \sum_{k=1}^{n} c(n,k)\, x^{n-k}$ (absolute order; $c(n,k)$ = (signless) Stirling #'s of 1st kind)

④ $F(\Pi_n, x) = \sum_{k=1}^{n} S(n,k)\, x^{n-k}$ (set partition lattice; $S(n,k)$ = Stirling #'s of 2nd kind)

Lattices:

DEF'N Say P is a meet semilattice if every $x,y \in P$ have some element $x \wedge y$ in P, called their meet, which is a greatest lower bound for x,y: any $z \leq x,y$ satisfies $z \leq x \wedge y \leq x,y$.



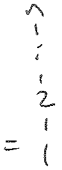
Note
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
- $x \wedge y = y \wedge x$
- $x \wedge x = x$
- $x \wedge y = x \iff x \leq y$

It is a join semilattice if $\forall\, x,y \in P$, $\exists$ a join $x \vee y$ in P, which is a least upper bound: any $z \geq x,y$ has $z \geq x \vee y \geq x,y$.

It is a lattice if it is both a meet and join semilattice.

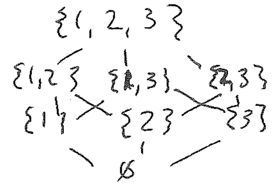
(Note: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.)

Examples:

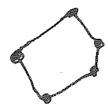
① Finite chains $[n] =$ 

are graded lattices. $F([n],x) = [n]_x = 1 + x + \cdots + x^{n-1}$

② finite Boolean lattices B_n are graded lattices

$B_3 =$ 

with $S \wedge T = S \cap T$, $S \vee T = S \cup T$, $\text{rank}(S) = |S|$. $F(B_n, x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

③ The pentagon lattice $P =$  is a lattice, but not graded.

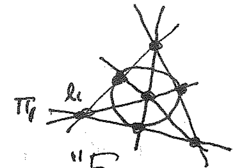
11/20

very useful! → ④ **Prop.** A finite meet semilattice $(P, \leq)$ always has a $\hat{0}$ (= minimum elt.) and if it has a $\hat{1}$ (= maximum elt.) then it is a lattice.

Proof: Check that $(\cdots((x_1 \wedge x_2) \wedge x_3) \cdots \wedge x_\ell)$ is a greatest lower bound for any non-empty, finite subset $\{x_1, x_2, \ldots, x_\ell\}$ in a meet semilattice.
Hence if $P = \{p_1, \ldots, p_\ell\}$ is a finite meet semilattices then $\hat{0} = p_1 \wedge \cdots \wedge p_\ell$ exists in P.
Also, if P has a $\hat{1}$, then given $x, y \in P$ the set $\{x_1, \ldots, x_\ell\}$ of all upper bounds for x, y (i.e., $x_i \geq x, y$) is nonempty (since $\hat{1}$ is in it), and one can check that $x_1 \wedge \cdots \wedge x_\ell = x \vee y$. ∎

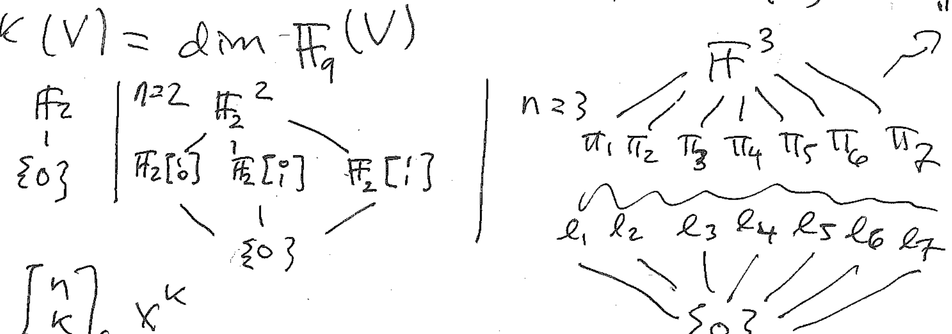
⑤ $B_n(q) = \mathcal{L}_n(q) = \mathcal{L}(\mathbb{F}_q^n) := \{\text{all } \mathbb{F}_q\text{-linear subspaces } V \subseteq \mathbb{F}_q^n\}$
= (finite) vecter space lattice

ordered by $\subset$ are graded lattices
with $V \wedge W := V \cap W$
$V \vee W := V + W \; (= \{v + w : v \in V, w \in W\})$
and $\text{rank}(V) = \dim_{\mathbb{F}_q}(V)$



"Fano Plane"

e.g. $q = 2$



$F(B_n(q), x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$

(6) $\Pi_n = \{$set partitions of $[n]\}$ ordered by refinement are graded lattices

with $\pi_1 \wedge \pi_2$ = common refinement of π_1, π_2

$\pi_1 \vee \pi_2$ = transitive closure of π_1's, π_2's blocks

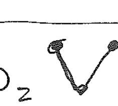
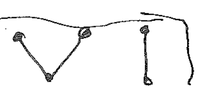
$\text{rank}(\pi) = n - \#\text{blocks}(\pi)$

e.g. $n=1$: 1 | $n=2$: 12 over 1\|2 | $n=3$: 123 over 1\|23, 12\|3, 13\|2 over 1\|2\|3

$$F(\Pi_n, x) = \sum_{k=1}^{n} S(n,k)\, x^{n-k}$$

(7) Given P, Q posets $P \sqcup Q$ = disjoint union having $p \in P$, $q \in Q$ incomparable

$P \times Q$ = (Cartesian) product w/ componentwise order: $(p_1, q_1) \le (p_2, q_2)$
$\iff p_1 \le_P p_2$ and $q_1 \le_Q q_2$

e.g. $P =$  $Q =$  $\Rightarrow P \sqcup Q =$ 

$P \times Q =$ 

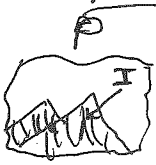
Prop. P, Q lattices, graded $\Rightarrow P \times Q$ lattice, graded

$F(P \times Q, x) = F(P, x) \cdot F(Q, x)$

dual poset: $P^* =$ 

(dual poset = flip $\le$ upside down)

(8) DEF'N An order ideal $I \subseteq P$ of a poset is a subset closed under going down: i.e., $p \in I$ and $p' \le p \Rightarrow p' \in I$.



$J(P) := \{$the lattice of all order ideals $I \subseteq P\}$ with ordered via $\subseteq$

$I_1 \wedge I_2 = I_1 \cap I_2$

$I_1 \vee I_2 = I_1 \cup I_2$

and $\text{rank}(I) = |I|$ (for $|P| < \infty$)

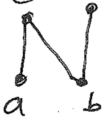
is a graded (for $|P| < \infty$) lattice

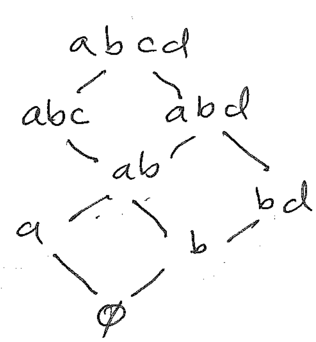
$F(J(P), x) = \sum_{\text{ideals } I \subseteq P} x^{|I|}$

It is in fact a distributive lattice, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(because $\cap$ and $\cup$ satisfy these relations)

e.g. $P =$ 

$J(P) =$ 

Prop $J(P \sqcup Q) = J(P) \times J(Q)$

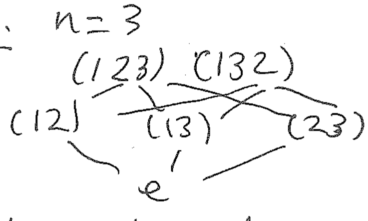
⑨ Three posets on $\mathfrak{S}_n$ that can be defined via transitive closure:

absolute order: trans. closure of $x<y$ when $x(i,j)=y$ and $cyc(x)>cyc(y)$ for some $1\le i<j\le n$

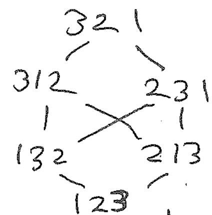
(strong) Bruhat order: trans. clo. of $x<y$ when $x(i,j)=y$ and $inv(x)<inv(y)$

(right) weak order: —''— of $x<y$ when $x(i,i+1)=y$ and $inv(x)<inv(y)$

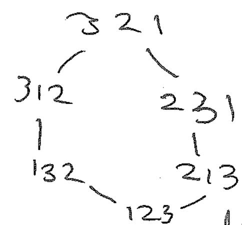
e.g. $n=3$



absolute order



Bruhat order



weak order

- All 3 are graded, with $rank(w)=n-cyc(w)$ for $\le_{abs}$ and $rank(w)=inv(w)$ for $\le_{Bruhat}, \le_{weak}$

$$F(\le_{abs},x)=\sum_{k=1}^{n} c(n,k)\,x^{n-k}$$

$$F(\le_{Bruhat},q)=F(\le_{weak},q)=\sum_{w\in\mathfrak{S}_n} q^{inv(w)}=[n]!_q$$

- Neither absolute order nor Bruhat are lattices, but weak order is a lattice (not obvious!)

11/22 ⑩ Dominance order on $\{\text{partitions } \lambda\vdash n\}$

$\mu \trianglelefteq \lambda$ if $\mu_1\le\lambda_1$

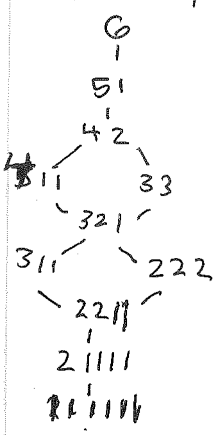
$(\mu_1,\mu_2,\ldots)$ $(\lambda_1,\lambda_2,\ldots)$ $\quad \mu_1+\mu_2\le\lambda_1+\lambda_2$

$\mu_1+\mu_2+\mu_3\le\lambda_1+\lambda_2+\lambda_3$

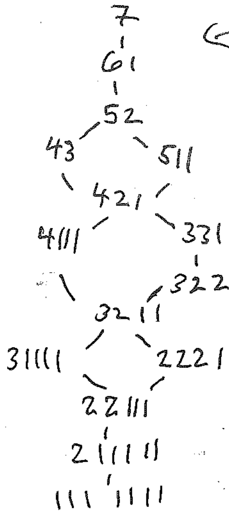
$\vdots$

For $n=1,2,3,4,5$ it is a total order, but not for $n\ge 6$:

← and not even graded for $n\ge 7$



$n=6$



$n=7$

Prop It is a lattice where if $\rho=\lambda\wedge\mu$, $\nu=\lambda\vee\mu$ then

$\rho_1+\cdots+\rho_k=\min(\lambda_1+\cdots+\lambda_k,\ \mu_1+\cdots+\mu_k)$

~~$\nu_1+\cdots+\nu_k=\max(\lambda_1+\cdots+\lambda_k,\ \mu_1+\cdots+\mu_k)$.~~

NO! this doesn't work for joins! can use duality tho.

Prop It is always self-dual (i.e. $P\cong P^{opp}=P^*$ (same poset elements, but $\le$ flipped)) via $\lambda\mapsto\lambda^t$ (transpose map).

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Distributive lattices (Stanley §3.4)

DEF'n/Prop In a lattice L,

(a) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ $\forall x, y \in L$

(b) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ $\forall x, y \in L$

and equality in (a) holds $\forall x, y \in L \Leftrightarrow$ equality in (b) holds $\forall x, y \in L$ in which case we call L distributive.

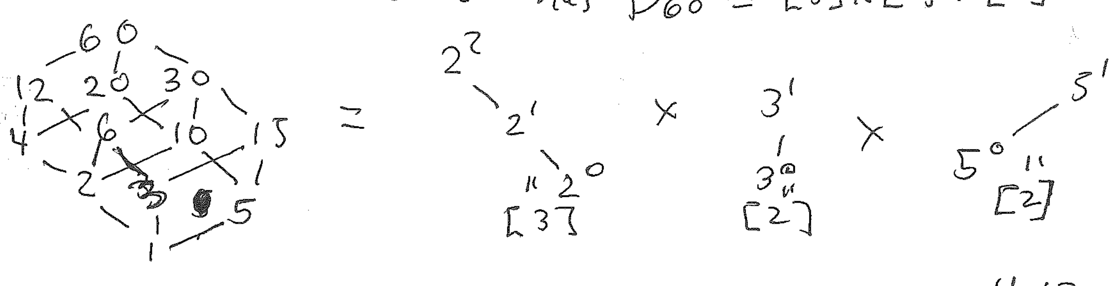
Examples ① For a poset P, $J(P) = \{\text{order ideals } I \subseteq P\}$ is a distr. lattice.

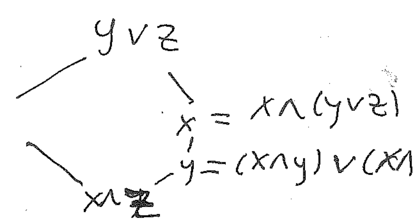
② L_1, L_2 distr. $\Rightarrow L_1 \times L_2$ distr.

③ The divisor poset $D_n = \{\text{all divisors of } n\}$ w/ $x \leq y$ if $x | y$ (for $n = 1, 2, \ldots$) is a distributive lattice, since if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for distinct primes p_i, then $D_n \cong [a_1 + 1] \times [a_2 + 1] \times \cdots \times [a_k + 1]$, and each chain is distributive.

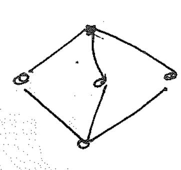
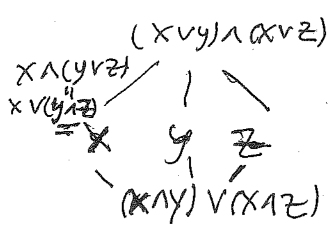
$d = p_1^{b_1} \cdots p_k^{b_k} \longmapsto (b_1 + 1, b_2 + 1, \ldots, b_k + 1)$

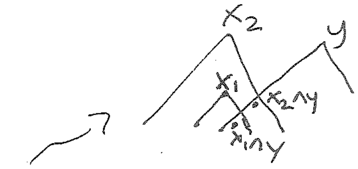
e.g. $n = 60 = 2^2 \cdot 3^1 \cdot 5^1$ has $D_{60} \cong [3] \times [2] \times [2]$.



④  is not distributive: 

(and dually ...)

⑤  is not distr., 



Proof of def'n/prop: Note that $x_1 \le x_2$ in $L \Rightarrow x_1 \wedge y \le x_2 \wedge y$

so $x \wedge (y \vee z) \ge \left. \begin{matrix} x\wedge y, \\ x \wedge z \end{matrix} \right\} \Rightarrow x \wedge (y \vee z) \ge (x \wedge y) \vee (x \vee z)$, proving part(a).

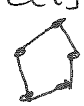
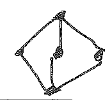
Part (b) follows dually (i.e., swapping $\le$ and $\ge$, and $\wedge$ and $\vee$).

Now assume that (a) holds w/ $=$ for all $x, y \in L$,
i.e. that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Then to prove (b) holds w/ $=$:

$$(x\vee y)\wedge(x\vee z) \overset{(a)}{=} \big((x\vee y)\wedge x\big) \vee \big((x \vee y)\wedge z\big) \quad \text{part (a) again}$$
$$= x \vee \big((x\wedge z)\vee(y\wedge z)\big)$$
$$= x \vee (y \wedge z). \checkmark$$

(Note $(x\vee y)\wedge x = x$ since $x \le x\vee y$, and similarly $x \vee (x \wedge z) = x$ since $x \ge x \wedge z$) ∎

Remark: G. Birkhoff (1948) showed that a lattice L is distr.
$\Longleftrightarrow$ L has no 5 element sublattice iso. to  or 

More importantly, he showed...

Thm (Birkhoff's fund. thm. of finite distributive lattices)
Every finite distributive lattice L is isomorphic to $J(P)$ (order ideal lattice)
for a poset P defined uniquely up to isomorphism,
namely $P \cong \mathrm{Irr}(L) := \{$the join irreducible $p \in L\}$

w/ the induced partial order as a subposet of L

($p = x_1 \vee \cdots \vee x_\ell$ for some $\{x_1, \ldots, x_\ell\} \subseteq L$
$\Downarrow$
$p = x_i$ for some i.)

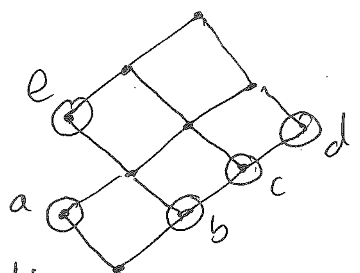
Remark: Fund. Thm. of fin. distr. lattices is a representation theorem. In the theory of lattices there are many representation thms. In fact, ...

We could have defined a lattice L abstractly ("universal algebra" approach) as a set L together w/ 2 binary operations $\vee, \wedge : L \times L \to L$ satisfying:

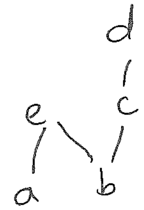
associativity	commutativity	idempotent	"absorbtion"
$x\wedge(y\wedge z) = (x\wedge y)\wedge z$	$x \wedge y = y \wedge x$	$x \wedge x = x$	$x \wedge (y \vee z)$
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \vee y = y \vee x$	$y \vee y = y$	$= x = x \vee (y \wedge z)$.

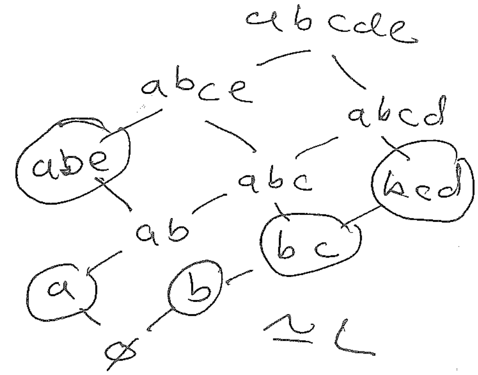
Then $\exists!$ partial order $\le$ on L s.t. $\wedge = \mathrm{glb}$, $\vee = \mathrm{lub}$,
(namely $x \le y$ iff $x \wedge y = x$ iff $y \vee x = y$)

Example of Birkhoff's FTFDL

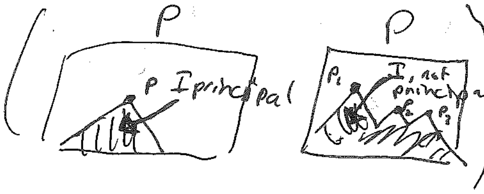
$L =$ 

is distributive, w/ elements of $P = Irr(L)$ labeled

$P = Irr(L) =$ 

$J(P) =$ 

$\cong L$



Note that the join-irreducibles in $J(P)$ = principal order ideals $I = P_{\leq p} = \{q \in P : q \leq p\}$

Proof of Birkhoff's Thm:

Given L finite and distributive, define maps

$$L \underset{g}{\overset{f}{\rightleftarrows}} J(P) \text{ where } P = Irr(L)$$

$$x \longmapsto f(x) := \{p \in Irr(L) : p \leq x\}$$

$$g(I) := p_1 \vee \cdots \vee p_\ell \longleftarrow I = \{p_1, \ldots, p_\ell\}$$

It's not hard to see both f, g order-preserving i.e., $x \leq y \Rightarrow f(x) \leq f(y)$, $I \subseteq I' \Rightarrow g(I) \leq g(I')$

We claim that in any finite lattice (not nec. distributive) one has $g(f(x)) = \bigvee_{\substack{p \in Irr(L): \\ p \leq x}} p = x$:

Certainly $\bigvee_{\substack{p \in Irr(L): \\ p \leq x}} p \leq x$ since each $p \leq x$, but also one can write $x = p_1 \vee p_2 \vee \cdots \vee p_\ell$ with each p_i join-irreducible, using downward induction on $x \in L$ (either $x \in Irr(L)$ or write $x = x_1 \vee x_2$ with $x_1 \lneq x$, $x_2 \lneq x$, and repeat.)

Hence indeed $x = \bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p = g(f(x))$.

On the other hand $f(g(I)) = \{q \in Irr(L) : q \leq p_1 \vee \cdots \vee p_\ell\} \supseteq I$, where $I = \{p_1, \ldots, p_\ell\}$

in a distributive lattice!
↓
but $q \le p_1 \vee \cdots \vee p_\ell \Rightarrow q = q \wedge (p_1 \vee \cdots \vee p_\ell)$

distributivity $\longrightarrow = (q \wedge p_1) \vee \cdots \vee (q \wedge p_\ell)$

$q \in \mathrm{Irr}(L) \longrightarrow \Rightarrow q = q \wedge p_i$ for some i

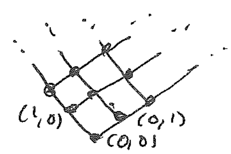
$\Rightarrow q \le p_i \in I$

I is an order ideal $\Rightarrow$ $\Rightarrow q \in I$

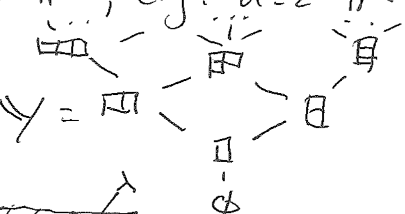
Hence $f(g(I)) = \{q \in \mathrm{Irr}(L) : q \le p_1 \vee \cdots \vee p_\ell\} \subseteq I$, and so $f(g(I)) = I$. ■

Remark: Certain ∞ distributive lattices are important...

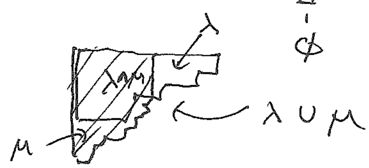
DEFN A finitary distributive lattice is a distr. lattice with a $\hat{0}$ which is locally finite (recall: all intervals are finite).

Examples: ① $\mathbb{N} =$  ② $\mathbb{N}^d$, e.g. $d=2$ $\mathbb{N}^2$ 

③ $\mathbb{Y}$ = Young's lattice on partitions

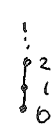
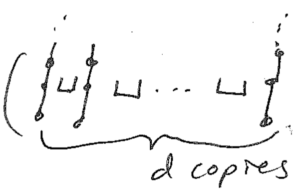
$\mathbb{Y} =$ 

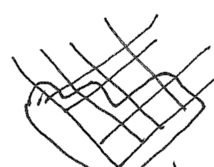
have $\mu \wedge \lambda = \mu \cap \lambda$
$\mu \vee \lambda = \mu \cup \lambda$



One can easily adapt argument to show this gen. of FTFDL!

Thm Every finitary distr. lattice L is isomorphic to
$J_f(P) := \{$all finite order ideals $I \subseteq P\}$
for some poset P having all principal order ideals $P_{\le p}$ finite, defined uniquely up to iso., namely $P \cong \mathrm{Irr}(L)$.

Examples: ① $\mathbb{N} =$  $= J_f($  $)$ ② $\mathbb{N}^d \cong J_f($  $)$

③ $\mathbb{Y} = J_f(\mathbb{N}^2 =$  $)$

can see visually how these finite order ideals correspond to partitions

Möbius inversion (Stanley §3.6, 3.7)

Let's reinterpret inclusion-exclusion as being about the poset $P = B_n = 2^{[n]}$ and functions $f = f_= : P \to R$ (a commutative ring), where we were given a new function

$$g = f_{\subseteq} : P \to R \text{ such that } g(S) = \sum_{T \subseteq S} f(T)$$

i.e. $g(y) = \sum_{x \in P} \zeta(x,y) f(x)$, where $\zeta(x,y) := \begin{cases} 1 & \text{if } x \leq y \text{ in } P \\ 0 & \text{else} \end{cases}$

and we could invert to get f via

$$f_=(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_{\subseteq}(T),$$

i.e., $f(y) = \sum_{x \in P} \mu(x,y) g(x)$ where $\mu(\overset{T}{x}, \overset{S}{y}) = \begin{cases} (-1)^{|y \setminus x|} & \text{if } x \leq y \text{ in } P \\ 0 & \text{else} \end{cases}$

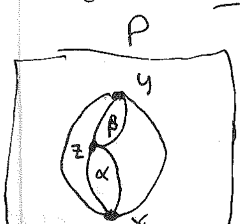
This same set-up works for other locally finite posets P, once we figure out what the $\zeta(x,y)$, $\mu(x,y)$ are, and where they live...

DEF'N The incidence algebra $I(P,R)$ of a loc. finite poset P (over a commutative ring R) is the ring of all functions

$$f : \underset{\{\text{intervals } [x,y] \text{ in } P\}}{\mathrm{Int}(P)} \longrightarrow R$$

with pointwise addition: $(\alpha + \beta)(x,y) = \alpha(x,y) + \beta(x,y)$

and convolution product: $(\alpha * \beta)(x,y) = \sum_{z \in [x,y]} \alpha(x,z)\beta(z,y)$ (finite sum)



and 2-sided identity element $\delta(x,y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$ ← Kronecker delta

---

We'll want to know that the zeta function
$\zeta(x,y) := \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{else} \end{cases}$ is always invertible in $I(P,R)$:

recall: group of units of R ↑

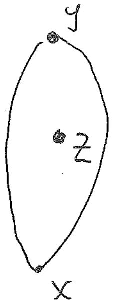
Prop: $\alpha \in I(P,R)$ has a (2-sided) inverse $\iff \alpha(x,x) \in R^\times \ \forall x \in P$.

Proof: $\alpha * \beta = \delta \iff (\alpha * \beta)(x,y) = \delta(x,y) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases} \ \forall x,y \in P$

$$(\alpha*\beta)(x,y) = \sum_{z \in [x,y]} \alpha(x,z)\beta(z,y)$$

which forces $\alpha(x,x)\beta(x,x) = 1$, so $\left\{ \begin{array}{l} \alpha(x,x) \in R^\times, \\ \beta(x,x) = \alpha(x,x)^{-1} \end{array} \right\} \forall x \in P$,

and then when $\alpha(x,x) \in R^\times$, the values for $\beta(x,y)$ are uniquely determined by induction on $\#[x,y]$ via



$$\alpha(x,x)\beta(x,y) + \sum_{z \in (x,y]} \alpha(x,z)\beta(z,y) = 0 \qquad (x,y] := \{z : x < z \leq y\}$$

$$\Rightarrow \beta(x,y) = -\alpha(x,x)^{-1} \cdot \sum_{z \in (x,y]} \alpha(x,z) \underbrace{\beta(z,y)}_{\#[z,y] < \#[x,y]}$$

Note that we can also get a left-inverse $\beta'(\cdot,\cdot)$ defined (recursively) by $\beta'(x,y) = -\alpha(y,y)^{-1} \sum_{z \in [x,y)} \beta'(x,z)\alpha(z,y)$ $\longrightarrow$ $[x,y) := \{z : x \leq z < y\}$

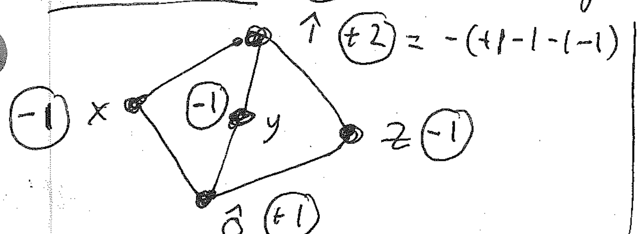
but then associativity of $*$ forces $\beta' = \beta' * (\alpha * \beta) = (\beta' * \alpha) * \beta = \beta$. ∎

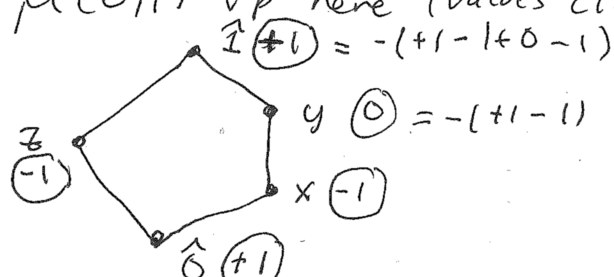
Cor $\zeta(\cdot,\cdot)$ has an inverse, called the Möbius function, $\mu = \zeta^{-1}$, defined recursively by $\boxed{\mu(x,x) = 1 \ \forall x \in P}$

and either $\mu(x,y) = -\sum_{z \in (x,z]} \mu(z,y) \quad \forall x < y$

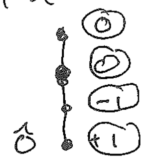
or $\boxed{\mu(x,y) = -\sum_{z \in [x,y)} \mu(x,z) \quad \forall x < y}$

Examples ① Let's compute $\mu(\hat{0}, p)$ $\forall p$ here (values circled)



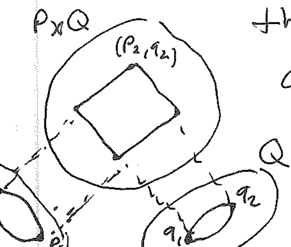


② In a finite chain, $\mu(x,y) = \begin{cases} +1 & \text{if } x = y \\ -1 & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases}$



③ **Prop:** In a product $P \times Q$, $\mu_{P\times Q}((p_1,q_1),(p_2,q_2)) = \mu_P(p_1,p_2)\,\mu_Q(q_1,q_2)$.

Proof: The function $\alpha(\cdot,\cdot) \in I(P\times Q, \mathbb{Z})$ defined by the RHS satisfies the correct initial condition and recurrence:



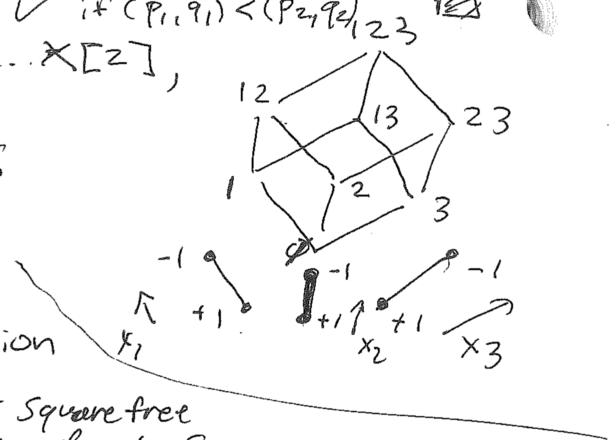
$\alpha((p,q),(p,q)) = \underbrace{\mu_P(p,p)}_{+1}\,\underbrace{\mu_Q(q,q)}_{+1} = +1$ ✓

$$\sum_{(p,q)\in[(p_1,q_1),(p_2,q_2)]} \mu_P(p_1,p)\,\mu_Q(q_1,q) = \Big(\underbrace{\sum_{p\in[p_1,p_2]} \mu_P(p_1,p)}_{=0 \text{ if } p_1<p_2}\Big)\Big(\underbrace{\sum_{q\in[q_1,q_2]} \mu_Q(q_1,q)}_{=0 \text{ if } q_1<q_2}\Big)$$

$= 0$ ✓ if $(p_1,q_1) < (p_2,q_2)$ ■

④ **Cor** In $B_n = 2^{[n]} \cong [2]^n = [2]\times[2]\times\cdots\times[2]$,

$\mu(T,S) = (-1)^{|S\setminus T|}$ for $T \subseteq S$

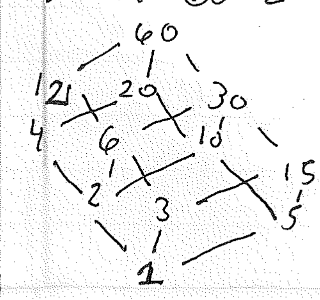


⑤ The number-theoretic Möbius function

$$\mu(n) := \begin{cases} (-1)^k & \text{if } n = p_1^1 \cdots p_k^k \text{ is square free with } k \text{ prime factors} \\ 0 & \text{if } n \text{ is not square free} \end{cases}$$

is really computing $\mu_{D_n}(d_1,d_2) = \mu\left(\frac{d_2}{d_1}\right)$ for $d_1 | d_2$ in the divisor poset $D_n \cong [a_1+1]\times[a_2+1]\times\cdots\times[a_k+1]$ when $n = p_1^{a_1}\cdots p_k^{a_k}$

e.g. $n = 60 = 2^2\cdot 3^1\cdot 5^1$



$\mu(3,12) = \mu\left(\frac{12}{3}\right) = \mu(4) = \mu(2^2) = 0$ ← not squarefree

$\mu(3,60) = \mu\left(\frac{60}{3}\right) = \mu(20) = \mu(2^2\cdot 5) = 0$

$\mu(2,60) = \mu\left(\frac{60}{2}\right) = \mu(30) = \mu(2^1\cdot 3^1\cdot 5^1) = (-1)^3 = 1$

12/4

Now let's state and use...

Thm (Möbius inversion formula)

If a poset P has all $P_{\leq p}$ finite, and $f, g : P \to R$ (a comm. ring) are related by $g(y) = \sum_{x \in P : x \leq y} f(x)$ $\forall y \in P$, then

$$f(y) = \sum_{x \in P : x \leq y} \mu(x,y)\, g(x) \quad \forall y \in P.$$

(And dually, if $P_{\geq p}$ are all finite, w/ $g(y) = \sum_{x : x \geq y} f(x)$, then $f(y) = \sum_{x : x \geq y} \mu(y,x)\, g(x)$.)

Proof: The free R-module $R^P := \{\text{functions } f : P \to R\}$ (w/ pointwise addition + scaling by elements of R) is actually a (right) $I(P,R)$-module, where $\alpha \in I(P,R)$ act on such f via $(f \cdot \alpha)(y) = \sum_{x \in P} f(x)\, \alpha(x,y)$.

Check that $(f \cdot \alpha) \cdot \beta = f(\alpha * \beta)$ since

$$((f \cdot \alpha) \cdot \beta)(y) = \sum_{x \in P} (f \cdot \alpha)(x)\, \beta(x,y)$$

$$= \sum_{x \in P} \sum_{x' \in P} f(x')\, \alpha(x',x)\, \beta(x,y)$$

$$= \sum_{x' \in P} f(x') \Big(\underbrace{\sum_{x \in P} \alpha(x',x)\, \beta(x,y)}_{(\alpha * \beta)(x,y)} \Big)$$

$$= (f \cdot (\alpha * \beta))(y) \checkmark$$

Then $g(y) = \sum_{x \in P, x \leq y} f(x) = \sum_{x \in P} f(x)\, \zeta(x,y)$,

i.e. $g = f \cdot \zeta$

$\Downarrow$ act on right by $\mu = \zeta^{-1}$

$g \cdot \mu = f$, i.e. $\sum_{x \in P} g(x)\, \mu(x,y) = f(y)$

$$\| $$

$$\sum_{x \in P, x \leq y} \mu(x,y)\, g(x).$$

☒

Cor 1 Inclusion-exclusion, for $P = B_n$.

Cor 2 (Number-theoretic Möbius inversion)

If $f, g : \mathbb{P} \to R$ (where $\mathbb{P} = \{1,2,3,\ldots\}$) are related by $g(n) = \sum_{d:d|n} f(d)$,

then $f(n) = \sum_{d:d|n} \underbrace{\mu\left(\frac{n}{d}\right)}_{=\mu(d,n) \text{ in } \underline{\text{divisor poset}}\ D_n} g(d)$

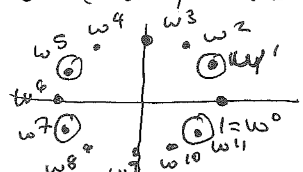
Examples

① Euler's phi-function (a.k.a. "totient function") $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$

$= |\{\bar{m} \in \mathbb{Z}/n\mathbb{Z} : \gcd(m,n) = 1\}|$

e.g. $\varphi(12) = 4 = |\{1,5,7,11\}|$

$= \#$ primitive n^{th} roots of unity in $\mathbb{C}$



It satisfies $f(n) = n = |\mathbb{Z}/n\mathbb{Z}| = \sum_{d:d|n} \varphi(d)$

$= \#\{n^{th}$ roots of 1 in $\mathbb{C}$ (not nec. prim.)$\}$; $\varphi(d) = \#\{$prim. d^{th} roots of 1 in $\mathbb{C}\}$

e.g. $\{0,1,\ldots,11\} = \underset{d=1}{\{0\}} \sqcup \underset{d=2}{\{6\}} \sqcup \underset{d=3}{\{4,8\}} \sqcup \underset{d=4}{\{3,9\}} \sqcup \underset{d=6}{\{2,10\}} \sqcup \underset{d=12}{\{1,5,7,11\}}$

Hence by Möbius inversion, $\varphi(n) = \sum_{d:d|n} \mu\left(\frac{n}{d}\right) f(d)$

$= \sum_{d|n} \mu\left(\frac{n}{d}\right) d$ (only care about squarefree $\frac{n}{d}$)

If $n = p_1^{a_1} \cdots p_k^{a_k}$ $\longrightarrow$ $= \sum_{\substack{S \subset \{1,2,\ldots,k\} \\ \{s_1,\ldots,s_\ell\}}} \mu(p_{s_1} p_{s_2} \cdots p_{s_\ell}) \frac{n}{p_{s_1} p_{s_2} \cdots p_{s_\ell}}$

$= \sum_{S \subset \{1,2,\ldots,k\}} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}$

$= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$

$= \prod_{i=1}^{k} \left(p^{a_i} - p^{a_i - 1}\right)$. ∎

② **Exercise:** Show that $f(n) := \sum_{\substack{\gamma \text{ primitive} \\ n^{th} \text{ root of} \\ \text{unity in } \mathbb{C}}} \gamma$ satisfies $f(n) = \mu(n)$ (number-theoretic Möbius fn.)

by checking that $\sum_{d|n} f(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$ (and *why* is this enough?).

③ *Skipped*: Can count *primitive necklaces* (+ give their generating function) via number-theoretic Möbius fn.

④ *P. Hall's (1936) application*: Given a finite group G, how to compute

$$f(G) := \#\{\text{subsets } A \subseteq G \text{ generating } G, \text{ i.e. } \langle A \rangle = G\}?$$

For a subgroup $H \leq G$, easy to compute

$$g(H) := \#\{\text{subsets } A \subseteq G \text{ generating some } K \leq H\}$$
$$= \#\{\text{subsets } A \subseteq H\} = 2^{|H|}.$$

But $g(H) = \sum_{K : K \leq H} f(K)$ ← in the lattice of subgroups $L(G)$

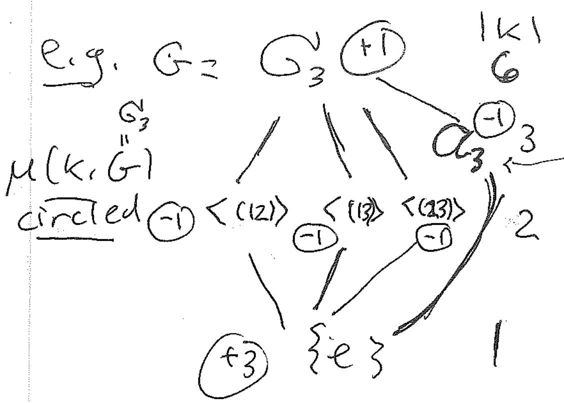
$H_1 \wedge H_2 = H_1 \cap H_2$

$H_1 \vee H_2 = \langle H_1 \cup H_2 \rangle$

So $f(H) = \sum_{K : K \leq H} \mu(K, H) g(K)$

$= \sum_{K : K \leq H} \mu(K, H) 2^{|K|}$, i.e. $\boxed{f(G) = \sum_{K \leq G} \mu(K, G) 2^{|K|}}$

e.g. $G = \mathfrak{S}_3$



← *alternating group* of even permutations (in this case $\mathfrak{A}_3 = \langle (123) \rangle = \langle (132) \rangle$)

So $f(\mathfrak{S}_3) = \sum_{K \leq G} \mu(K, \mathfrak{S}_3) 2^{|K|}$

$= 2^6 - (2^2 + 2^2 + 2^2 + 2^3) + 3 \cdot 2^1$

$= 64 - 20 + 6$

$= 50.$

Computing Möbius functions (§3.8, 3.9 Stanley)

Let's develop some tools for computing Möbius functions of lattices, and apply them to lattices we like: Π_n, $\mathcal{L}_n(q)$, $J(P)$

Another useful algebraic tool:

DEF'N: For a lattice L, its Möbius algebra $A(L, \mathbb{K})$, over a field $\mathbb{K}$, is $\mathbb{K}^L$ with a $\mathbb{K}$-basis $\{f_x\}_{x \in L}$ that multiplies by the rule: $f_x f_y = f_{x \wedge y}$
(= semigroup alg. for $\wedge$ on L)

Prop. For a finite lattice L, there is a ring isomorphism

$$A(L, \mathbb{K}) \xrightarrow{\varphi} \mathbb{K}^{|L|} := \{ \underbrace{\mathbb{K} \times \cdots \times \mathbb{K}}_{|L| \text{ times}} \text{ w/ } \mathbb{K}\text{-basis } \{e_x\}_{x \in L}$$

$$f_y \longmapsto \sum_{x \leq y} e_x$$

multiplying as orthogonal idempotents: $e_x^2 = e_x$, $e_x e_y = 0$ if $x \neq y$ $\}$.

We have $\delta_y := \varphi^{-1}(e_y) = \sum_{x \leq y} \mu(x,y) f_x$, so $f_y = \sum_{x \leq y} \delta_x$.

Hence $\{\delta_y\}_{y \in L}$ are a $\mathbb{K}$-basis of orthogonal idempotents in $A(L, \mathbb{K})$.

Proof: φ is a $\mathbb{K}$-vector space iso. since its matrix is unippertriangular

$$\varphi = {}_{f_y}\Big\{ \overbrace{\begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}}^{e_x}$$

for any linear ordering of L that extends $\leq$.

Also can check $\varphi(f_y f_z) = \varphi(f_{y \wedge z}) = \sum_{x \leq y \wedge z} e_x$

$$\varphi(f_y)\varphi(f_z) = \Big(\sum_{x \leq y} e_x\Big)\Big(\sum_{w \leq z} e_w\Big) = \sum_{\substack{(x,w): \\ x \leq y,\, w \leq z}} e_x e_w = \sum_{\substack{x \leq y, \\ x \leq z}} e_x = \sum_{x \leq y \wedge z} e_x. \checkmark$$

The fact that $\varphi^{-1}(e_y) = \sum_{x \leq y} \mu(x,y) f_x$ follows from

$f_y = \sum_{x \leq y} \varphi^{-1}(e_x)$ via Möbius inversion. $\blacksquare$

Cor (Weisner's Thm)

If $a \neq \hat{1}$ in a finite lattice L, then $\sum_{x:\, a \wedge x = \hat{0}} \mu(x, \hat{1}) = 0$.

(Dually, if $a \neq \hat{0}$, then $\sum_{x:\, a \vee x = \hat{1}} \mu(\hat{0}, x) = 0$.)

Proof: Compute in 2 ways:

$$\left(\sum_{b \leq a} \delta_b\right) \delta_{\hat{1}} = f_a \delta_{\hat{1}} = f_a \cdot \left(\sum_{x \leq \hat{1}} \mu(x, \hat{1}) f_x\right) = \sum_{x \in L} \mu(x, \hat{1}) f_{a \wedge x}.$$

$\left(\sum_{b \leq a} \delta_b\right) \delta_{\hat{1}} = 0$, since $b \leq a \Rightarrow b \neq \hat{1}$

extract coeff. of $\hat{0}$:

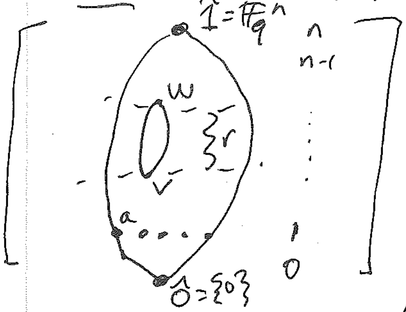
$$0 = \sum_{x:\, a \wedge x = \hat{0}} \mu(x, \hat{1}).$$

∎

Examples:

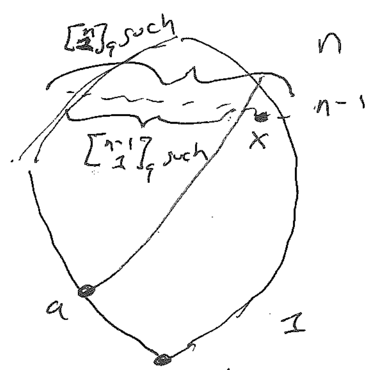
① **Prop:** In $\mathcal{L}_n(q)$, $\mu(\hat{0}, \hat{1}) = (-1)^n q^{\binom{n}{2}}$, and hence $\mu(V, W) = (-1)^r q^{\binom{r}{2}}$ if $\dim(W/V) = r$.

Picture →



Proof: Pick a line a, and then

$$0 = \sum_{x:\, a \vee x = \hat{1}} \mu(\hat{0}, x)$$



$$\mu(\hat{0}, \hat{1}) = -\sum_{\substack{x \lneq \hat{1},\\ a \vee x = \hat{1}}} \mu(\hat{0}, x)$$

count # x of dim $= n-1$ s.t. $a \not\leq x$

$a \vee x = \hat{1}$ ← forces x to have $\dim = n-1$ since $\dim(\underbrace{x+a}_{x \vee a}) = \dim(x) + \underbrace{\dim(a)}_{1} - \dim(\underbrace{x \cap a}_{x \wedge a}) \leq \dim(x) + 1$

$$= -\left(\left[{n \atop 1}\right]_q - \left[{n-1 \atop 1}\right]_q\right) \mu_{\mathcal{L}_{n-1}(q)}(\hat{0}, \hat{1})$$

$$= -\left((1+q+\cdots+q^{n-1}) - (1+q+\cdots+q^{n-2})\right) \cdot \mu_{\mathcal{L}_{n-1}(q)}(\hat{0}, \hat{1})$$

$$= -q^{n-1} \mu_{\mathcal{L}_{n-1}(q)}(\hat{0}, \hat{1}) \underset{\text{iterate}}{=} (-1)^n q^{(n-1)+(n-2)+\cdots+2+1+0}$$

$$= (-1)^n q^{\binom{n}{2}}.$$

∎

(2) This argument generalizes ...

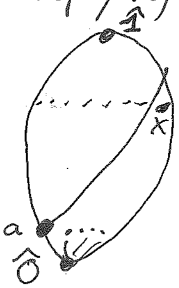
DEF'N A <u>graded</u> lattice L is (<u>upper-</u>) <u>semimodular</u> if

$$\text{rank}(x \vee y) + \text{rank}(x \wedge y) \leq \text{rank}(x) + \text{rank}(y) \quad \forall x, y \in L.$$

<u>e.g.</u> (finitary) distributive lattices, $\mathcal{L}_n(q)$, Π_n (Exercise!)
These two are <u>modular</u>; have $=$ $\forall x, y$ above

<u>Prop.</u>: L finite and upper-semimodular $\Rightarrow$ $\mu(\cdot,\cdot)$ <u>alternates in sign</u>,
i.e. $(-1)^{\text{rank}(y) - \text{rank}(x)} \mu(x,y) \geq 0$.

<u>Proof</u>: WLOG, $x = \hat{0}$, $y = \hat{1}$ and pick any <u>atom</u> $a > \hat{0}$.
to apply Wiesner to, giving $0 = \sum_{x : x \vee a = \hat{1}} \mu(\hat{0}, x)$



rank($\hat{1}$)
rank($\hat{1}$)-1

$$\mu(\hat{0},\hat{1}) = -\sum_{\substack{x \lneq \hat{1}: \\ x \vee a = \hat{1}}} \mu(\hat{0}, x)$$

$\mu(\hat{0},x)$ has sign $(-1)^{r(\hat{1})-1}$ by <u>induction</u>

$\left.\begin{matrix} x \lneq \hat{1}: \\ x \vee a = \hat{1} \end{matrix}\right\} \Rightarrow$ forces x to be of rank $r(\hat{1}) - 1$ by upper-semimodularity
$r(x \vee a) \leq r(x) + r(a) - r(x \wedge a)$
$\leq r(x) + 1$

$$\Rightarrow (-1)^{r(\hat{1})} \mu(\hat{0},\hat{1}) \geq 0.$$

∎

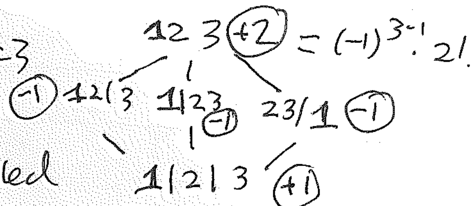
(3) We could similarly use Wiesner to compute $\mu_{\Pi_n}(\hat{0},\hat{1})$, but instead let's use <u>Möbius inversion</u>...

<u>Prop</u> In Π_n ("set partition lattice"), $\sum_{\pi \in \Pi_n} \mu(\hat{0},\pi)\, t^{\#\text{blocks}(\pi)} = t(t-1)(t-2)\cdots(t-(n-1))$
$= \sum_{k=1}^{n} s(n,k)\, t^k$

coeff. of t^{n-1} ⇝

$$\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$$

<u>e.g.</u> $n=3$



$\mu(\hat{0},\pi)$ circled

$$t^3 - 3t^2 + 2t^1 = t(t-1)(t-2).$$

Proof: It suffices to prove it for $t \in \{1,2,3,\ldots\}$, which we do by computing in two ways $\chi(K_n, t) = \#\{$<u>proper</u> vertex t-colorings of $K_n\}$

$$= t\,(t-1)\,(t-2)\cdots(t-(n-1))$$

(color 1, then color 2, etc.)

$$= \#\{\text{vertex } t\text{-colorings } c \text{ of } K_n \text{ whose associated } \underline{\text{color partition}}\ \pi(c) = \hat{0}\}$$

($\hat{0} = \{\{1\},\{2\},\ldots,\{n\}\}$)

If we define $f, g : \Pi_n \to \mathbb{Z}$ by

$$f(\pi) = \#\{\text{vertex } t\text{-colorings } c \text{ of } K_n \text{ having } \pi(c) = \pi\}$$

$$g(\pi) = \#\{\text{—— " ——}\ \pi(c) \geq \pi\} = \sum_{\tau : \tau \geq \pi} f(\tau)$$

($\geq$: coarsens)

$$= t^{\#\text{blocks}(\pi)} \quad \text{(since can color each block of } \pi \text{ independently)}$$

then by Möbius inversion $f(\pi) = \sum_{\tau \geq \pi} \mu(\pi, \tau)\, g(\tau) = \sum_{\tau \geq \pi} \mu(\pi, \tau)\, t^{\#\text{blocks}(\tau)}$.

So $\chi(K_n, t) = f(\hat{0}) = \sum_{\tau \geq \pi} \mu(\hat{0}, \tau)\, t^{\#\text{blocks}(\tau)}$. ∎

Remark: This determines $\mu(\pi, \tau)$ for all $\pi, \tau \in \Pi_n$ as follows:

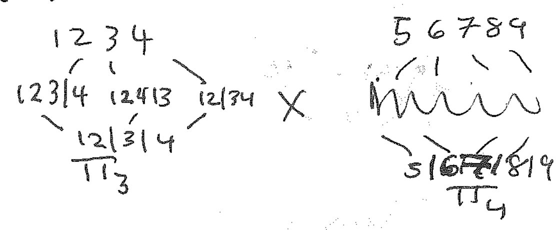
If τ has blocks $S_1, \ldots, S_\ell$ and π refines these into $n_1, \ldots, n_\ell$ blocks respectively,

then $[\pi, \tau]_{\Pi_n} \cong \Pi_{n_1} \times \Pi_{n_2} \times \cdots \times \Pi_{n_\ell}$

So $\mu_{\Pi_n}(\pi, \tau) = (-1)^{n_1}(n_1 - 1)! \cdots (-1)^{n_\ell}(n_\ell - 1)!$

e.g. $\tau = 1234 \,\|\, 56789$
$\pi = 12|3|4 \,\|\, 5|67|8|9$
$n_1 = 3 \qquad n_2 = 4$

$\Rightarrow [\pi, \tau] \cong$



To compute μ for distributive lattices $J(P)$, let's introduce another useful lemma:

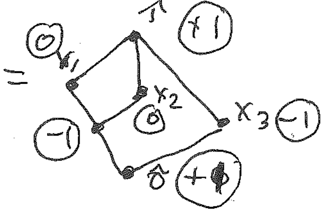
Thm (Rota's Crosscut Thm)

In a finite lattice L, w/ coatoms $\{x_1, \ldots, x_\ell\}$ (= elts. $x \lessdot \hat{1}$), we have

$$\mu(\hat{0}, \hat{1}) = \sum_{\substack{S \subseteq \{x_1, \ldots, x_\ell\} \\ \wedge S = \hat{0}}} (-1)^{|S|}.$$

In particular, $\mu(\hat{0},\hat{1}) = 0$ if $\hat{0}$ is not a meet of coatoms (or if $\hat{1}$ is not a join of atoms).

e.g. $L =$



$(\mu(\hat{0}, x))$

S	$(-1)^{\|S\|}$
$\{1,3\}$	$+1$
$\{2,3\}$	$+1$
$\{1,2,3\}$	-1
	$+1 = \mu(\hat{0},\hat{1})$ ✓

pf: In the Möbius algebra $A(L,K)$, compute in 2 ways:

$$\sum_{S \subseteq \{x_1, \ldots, x_\ell\}} (-1)^{|S|} \prod_{x_i \in S} f_{x_i} = \prod_{i=1}^{\ell} (f_{\hat{1}} - f_{x_i}) = \prod_{i=1}^{\ell} \Big(\sum_{y \not\leq x_i} \delta_y \Big)$$

$$\sum_{S \subseteq \{x_1, \ldots, x_\ell\}} (-1)^{|S|} f_{\wedge S} \qquad\qquad = \sum_{\substack{(y_1, \ldots, y_\ell): \\ y_i \not\leq x_i}} \delta_{y_1} \cdots \delta_{y_\ell} = \sum_{y \not\leq x_i \forall i} \delta_y = \delta_{\hat{1}}$$

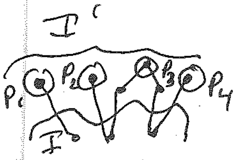
(since every $y \neq \hat{1}$ lies below some coatom.)

extract coeff. of $f_{\hat{0}}$ ⟹ Theorem ✓.

Cor In a finite (stanley) distributive lattice $L = J(P)$,

$$\mu(I, I') = \begin{cases} (-1)^{|I' \setminus I|} & \text{if } I' \setminus I \text{ is an \underline{antichain} in } P \\ 0 & \text{otherwise.} \end{cases}$$

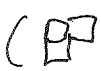
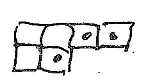
Proof: Check that the coatoms of $[I, I']$ are $x_i = I' \setminus \{p_i\}$ for maximal elts of $I' \setminus I$.

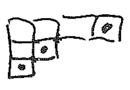


So their meet $x_1 \wedge \cdots \wedge x_\ell = I' \setminus \{p_1, \ldots, p_\ell\} = I$

$\iff$ every elt. of $I' \setminus I$ is maximal, i.e. $I' \setminus I$ is an antichain! ■

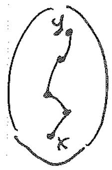
Example: In Young's lattice $\mathbb{Y}$, $\mu(\lambda, \rho) = \begin{cases} (-1)^{|\rho/\lambda|} & \text{if } \rho/\lambda \text{ has no 2 boxes in same row or col.} \\ 0 & \text{otherwise.} \end{cases}$

e.g. $\mu($  $,$  $) = 0$

$\mu($  $,$  $) = (-1)^3 = 1.$ ✓

Connection of Möbius function to topology; (skipped!)

Prop: (P. Hall's Thm) $\mu(x,y) = \sum_{\substack{\text{chains} \\ x = x_0 < x_1 < \cdots < x_\ell = y}} (-1)^\ell$



Pf: Call the RHS $\mu'(x,y)$ and check that

$$\sum_{z : x \leq z \leq y} \mu'(x,z) \overset{?}{=} \begin{cases} 1 & \text{if } x = y \quad \text{(easy ✓)} \\ 0 & \text{if } x < y \end{cases}$$

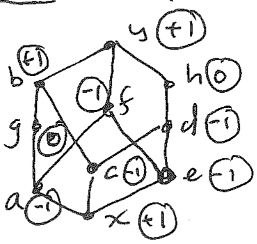
$$\sum_{(z,\, x_0 < \cdots < x_\ell)} (-1)^\ell \underset{\text{if } x<y}{=} 0$$ via a sign-reversing involution that adds/removes y from end of the chain. ▨

(where $x_0 = x$, $x_\ell = z$)

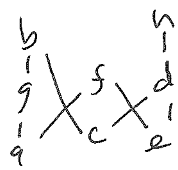
Note: $x_1 < \cdots < x_{\ell-1}$ is a chain in the open interval $(x,y) := \{ z \in P : x < z < y \}$.

DEF'N The order complex of a poset Q is the abstract simplicial complex $\Delta Q \subseteq 2^Q$ on vertex set Q w/ faces (simplices) F = chains in Q.

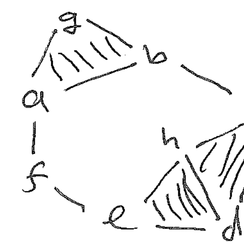
e.g. $\mu(x,z)$



$\rightsquigarrow (x,y) =$



$\rightsquigarrow \Delta(x,y) =$



P. Hall's thm says

Prop: $\mu(x,y) = \sum_{\text{faces } F \text{ of } \Delta(x,y)} (-1)^{\dim F} =: \tilde{\chi}(\Delta(x,y)) \overset{\text{Euler-Poincaré Thm}}{=} \sum_{i=-1}^{\infty} (-1)^i \dim_{\mathbb{K}} \tilde{H}_i(\Delta(x,y), \mathbb{K})$

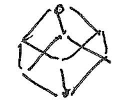
(reduced) Euler characteristic; (reduced) homology groups

e.g. above $\mu(x,y) = \tilde{\chi}\Big($  $\Big) = \tilde{\chi}\big(\text{circle } S^1 \big) = (-1)^1$ since $\tilde{H}_i(S^1, \mathbb{K}) = \begin{cases} 0 & i = -1 \\ 0 & i = 0 \\ 1 & i = 1 \\ 0 & i \geq 2 \end{cases}$

(homotopy invariance of Euler characteristic)

Rmk: For this reason, graded posets P w/ $\mu(x,y) = (-1)^{r(y)-r(x)}\ \forall x, y \in P$ are called Eulerian posets.

e.g. face lattices $F(P)$ of convex polytopes P are Eulerian. //

$P =$  $= n$-simplex, $F(P) =$  $= B_{n+1}$

/11

More fun (algebraic/enumerative) combinatorics:

- tableaux world:

1	2	4	6
3	5		
7			

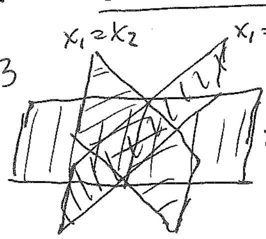
  - You'll learn all about this if you take 8669 in the spring
  - connections to:
    - representation theory of S_n, symmetric group
    - representation thry of GL_n, general linear gp.
    - cohomology of Grassmannian $Gr(k,n)$
    - "other types", i.e. Type A, B, C, D, ... Lie algebras
- More about posets:
  - Sperner theory of posets:
    - Dilworth's theorem / Greene-Kleitman invariants
    - Sperner's theorem / LYM inequality
    - Peck posets, symmetric chain decompositions, ...
  - Dynamics on posets:
    - "promotion" and "evacuation"
  - Eulerian posets (graded posets P w/ $\mu(x,y) = (-1)^{r(y)-r(x)}$ $\forall x,y \in P$)
    - "face enumeration" and the cd-index
    - behave like (and include) face lattice $F(P)$ of convex polytope P
- discrete geometry:
  - <u>hyperplane arrangements</u> (very related to posets/lattices)
  - simplicial complexes / convex polytopes
  - matroids

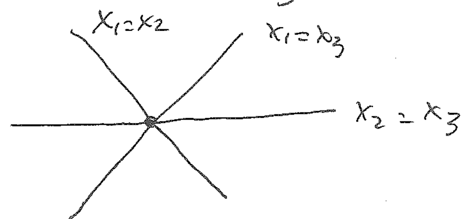
Hyperplane arrangements + posets (a very brief! survey) (See Stanley Ch. 3.1), or his monograph on hyp. arr's)

For a field $\mathbb{K}$, a hyperplane arrangement $\mathcal{A}$ is a finite! collection $\mathcal{A} = \{H\}$ of (affine) hyperplanes in $\mathbb{K}^n$ (i.e., codimension one subspaces of $\mathbb{K}^n$).

E.g. The Braid arrangement $\mathcal{B}_n := \{x_i = x_j : 1 \le i < j \le n\}$

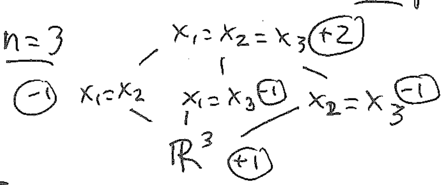
$n=3$

$\mathcal{B}_3 =$ 

" = " 

DEF'N The intersection poset of $\mathcal{A}$ $L(\mathcal{A},\mathbb{K})$ is the poset of non-empty intersections of hyperplanes in $\mathcal{A}$, ordered by reverse inclusion. It is a (finite) meet semilattice, graded by co-dimension. Its characteristic polynomial is
$$\chi_{(\mathcal{A},\mathbb{K})}(t) = \sum_{x \in L(\mathcal{A},\mathbb{K})} \mu(\hat{0},x)\, t^{\dim(x)}$$
$\hat{0} \in L(\mathcal{A},\mathbb{K}^n) = \mathbb{K}^n$, the "empty intersection"

E.g. $L(\mathcal{B}_n) = \Pi_n$, set partition lattice, so $\chi_{\mathcal{B}_n}(t) = t(t-1)(t-2)\cdots(t-(n-1))$.

$n=3$ 

$\chi_{\mathcal{B}_3}(t) = t^3 - 3t^2 + 2t = t(t-1)(t-2)$.

Thm (Zaslavsky's Thm) For a hyperplane arr. $\mathcal{A}$ in $\mathbb{R}^n$, the # of regions of $\mathcal{A}$ (= con. components of $\mathbb{R}^n \setminus \mathcal{A}$) is $r(\mathcal{A}) = |\chi_{\mathcal{A}}(-1)|$.

E.g. $r(\mathcal{B}_n) = |(-1)(-1-1)(-1-2)\cdots(-1-(n-1))| = n!$ $\quad (x_{\pi(1)} > x_{\pi(2)} > \cdots > x_{\pi(n)}\ \forall \pi \in \mathfrak{S}_n)$

$n=3$  = 6 regions! ($x_1 > x_2 > x_3$, etc...)

Thm (Finite field method) For q a prime power, and $\mathcal{A}$ arr. in $\mathbb{F}_q^n$
$\#\{v \in \mathbb{F}_q^n : v \notin H\ \forall H \in \mathcal{A}\} = \chi_{\mathcal{A}}(q)$ (and if $\mathcal{A}$ defined over $\mathbb{Q}$ for ∞-many q, $L(\mathcal{A},\mathbb{Q}) \simeq L(\mathcal{A},\mathbb{F}_q)$)

E.g. $\chi_{\mathcal{B}_n}(q) = q(q-1)\cdots(q-(n-1)) = \chi_{K_n}(q) = \#\{\text{proper } q\text{-colorings of complete graph } K_n\}$.

Thm (Orlik-Solomon alg.) For $\mathcal{A}/\mathbb{C}$, cohomology ring $H^*(\mathbb{C} \setminus \mathcal{A})$ is determined by $L(\mathcal{A},\mathbb{C})$.

Remark: $\pi_1(\mathbb{C}^n \setminus \mathcal{B}_n) =$ pure braid group, explaining the name.