Graph coloring,
UMTYMP Advanced Topics, Fall 2020

Recall that for a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the smallest number of colors needed to (properly) color the vertices of G. Graphs with $\chi(G) = 2$ are called *bipartite*.

If G has a subgraph isomorphic to K_d, the complete graph on d vertices, then $\chi(G) \geq d$ because it requires d colors to color even that subgraph.

1. For each $d \geq 3$, give an example of a graph G which does not contain a subgraph isomorphic to K_d but with $\chi(G) \geq d$.

Remark: In fact, much more is true. The *girth* of a graph G is the size of the smallest cycle in G. A classic result of Erdös (probably beyond what we'll prove in this class) says that for any g, d, there exists a graph G with girth $\geq g$ and $\chi(G) \geq d$.

Let $\Delta(G)$ denote the maximum degree of G. We saw a simple proof by induction that $\chi(G) \leq \Delta(G) + 1$.

2. Show that the bound just mentioned is sharp: for each $d \geq 1$, give an example of a graph with $\Delta(G) = d$ and $\chi(G) = d + 1$. How many examples can you think of?

Remark: *Brooks' theorem* says that the only G with $\chi(G) = \Delta(G) + 1$ are the "obvious" examples.

3. Let G be a bipartite graph on n vertices. How big can $\Delta(G)$ be?

4. For each $d \geq 1$, give an example of a bipartite graph G for which the *minimum degree* of G is d.

The *chromatic polynomial* $\chi_G(k)$ of G is the polynomial in k which **counts** (proper) k-colorings:

$$\chi_G(k) = \#k\text{-colorings of } G.$$

For example, if G has n isolated vertices (no edges), then $\chi_G(k) = k^n$. On the other hand if $G = K_3$ is the triangle, then $\chi_G(k) = k(k-1)(k-2)$ (**think about this**!).

5. What is the chromatic polynomial $\chi_{K_n}(k)$ of the complete graph K_n?
6. Let T be a tree on n vertices. What is $\chi_T(k)$?
7. Let C_n be the cycle graph on n vertices. What is $\chi_{C_n}(k)$? (**This one is harder...**)
8. Can you see why $\chi_G(k)$ is always a **polynomial** in k?