Math 4990: Ramsey Theory (+ the probabilistic method)

12/8

Ch. 13 + 15 of Bóna

Reminders: • HW #5 should be graded + returned soon, if not already...

• The final exam has been posted, due in 1 week on 12/15

For the last two classes we'll discuss fun/'cultural' topics that won't be evaluated...

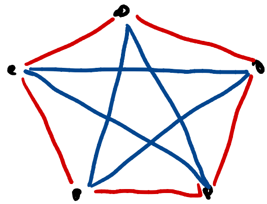
Today the topic is Ramsey theory, an important area of modern combinatorics.

Ramsey theory starts from the following curious observation: in any group of 6 people, there are 3 people who all pairwise know each other, or 3 people who pairwise don't know each other.

Note that 5 people would not suffice for this.

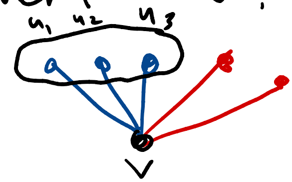
Indeed, if we represent 'know each other' by a red edge, and 'don't know each other' by a blue edge, then

Consider the following edge-coloring of K_5:



It has no blue triangle or red triangle! But how do we show that there is a blue Δ or red Δ in every red-blue edge coloring of K_6?

Consider any vertex v:



Since $\deg(v) = 5$, of the edges leaving v, we have at least 3 that are the same color, say blue WLOG.

Look at these 3 blue edges from v and the vertices u_1, u_2, u_3 they connect to. If any of the edges between u_1, u_2, u_3 are blue, this edge together w/ two of the blue edges from v gives a blue Δ.

Otherwise, all the edges between u_1, u_2, u_3 are red and they form a red Δ. Tada! //

Ramsey's theorem is the extension of this problem beyond triangles (i.e., 3 people):

Thm (Ramsey's Theorem) For any $n \geq 2$, $\exists$ a smallest number $R(n)$ (the "Ramsey number") such that in any red-blue edge coloring of K_N, w/ $N \geq R(n)$, there is some monochromatic (i.e., all blue or all red) K_n-subgraph.

---

e.g. We saw that $R(3) = 6$ above.

Ramsey theory studies results like Ramsey's theorem. The tagline of Ramsey theory is:

"any sufficiently large system has a big subsystem that is <u>ordered</u>,"

or more succinctly

"<u>complete</u> disorder is impossible."

How to prove Ramsey's theorem? The same kind of inductive argument like w/ the case $n=3$ will work, but we need to use 2-parameter/asymmetric Ramsey #'s:

$R(k,l) :=$ minimum $R(k,l)$ s.t. in any red-blue edge coloring of K_N, w/ $N \geq R(k,l)$, there is either a red K_k subgraph or a blue K_l subgraph.

Note: $R(n,n) = R(n)$ in our previous notation.

Note: $R(k,l) = R(l,k)$, by symmetry.

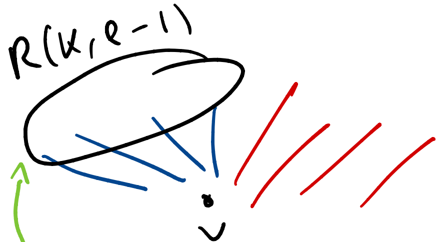
Also: $R(k,2) = R(2,k) = k$ (since any blue edge will give a blue K_2).

Prop: $R(k,l) \leq R(k,l-1) + R(k-1,l)$

So in particular, $R(k,l)$ exists! and so does $R(n) = R(n,n)$, proving Ramsey's theorem, bearing in mind $R(k,2)$, $R(2,l)$ are base cases of the induction.

Pf: Let $N := R(k, \ell-1) + R(k-1, \ell)$, and consider any red-blue edge coloring of K_N.

Let v be any vertex. Since $\deg(v) = N-1 = R(k, \ell-1) + R(k-1, \ell) - 1$, by Pigeonhole Principle either $R(k, \ell-1)$ of edges leaving v are blue, or $R(k-1, \ell)$ are red:



Assume WLOG $R(k, \ell-1)$ of them are blue, and focus on the $K_{R(k,\ell-1)}$-subgraph of those vertices. By definition of $R(k, \ell-1)$, either we have a red K_k here ... in which case we win!

Or we have a blue $K_{\ell-1}$ here, which we can combine with v to get a blue K_ℓ and then we win! again. $\boxtimes$

Same inductive argument gives upper bound for the Ramsey numbers:

Prop: $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$

Pf: Base case $R(k,2) = k = \binom{k}{k-1}$. ✓

Induction, we saw that

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell)$$
$$\text{(induction)} \leq \binom{k+\ell-3}{k-1} + \binom{k+\ell-3}{k-2} = \binom{k+\ell-2}{k-1}.$$

$\blacksquare$

Prop: $R(n) = R(n,n) \leq 4^{n-1}$

Pf: $R(n,n) \leq \binom{2(n-1)}{n-1}$ by above

and $\leq 4^{n-1}$

simple application of e.g. Stirling's formula

$\blacksquare$

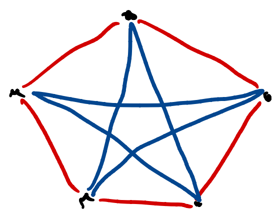
But note this bound is far off in case $n=3$ we saw:

$$6 = R(3) \leq 4^{3-1} = 4^2 = 16.$$

Question: How can we find a good lower bound for the Ramsey numbers?

In other words, how can we find a coloring of edges of big K_N w/out monochromatic K_n?

Recall for $n=3$ we had coloring:



Neat idea: Use randomness to find a good edge coloring for our purposes.

This is called the probabilistic method.

Thm (Erdős)
$$R(n) = R(n,n) \geq 2^{n/2}$$

Pf: Let $N \leq 2^{n/2}$, and consider coloring edges of K_N red + blue randomly, e.g., by flipping a coin for each edge.

We want to show that

$$\Pr(\text{there is } \underline{\text{no}} \text{ monochromatic } K_n) > 0,$$

which proves that some coloring must have no mono. K_n, although we have no idea what it looks like!

How to show $\Pr > 0$? We'll show:

$$\Pr(\text{there is } \underline{\text{some}} \text{ mono. } K_n) < 1$$

How to do this? First observe that for any K_n-subgraph H of K_N,

$$\Pr(H \text{ is mono } {}^{\text{blue or}}_{\text{red}}) = \frac{2 \to \text{all blue or all red}}{2^{\binom{n}{2}} \to \text{total \# colorings}}$$

$$\Pr(\text{some } H \text{ is mono.}) \le \sum_H \Pr(H \text{ is mono})$$

think about $\#(A \cup B) \le \#A + \#B$

$$= \binom{N}{n} 2^{1-\binom{n}{2}}$$

$$< \frac{N^n}{n!} 2^{1-\binom{n}{2}}$$

since $N \le 2^{n/2}$

easy to show by induction

$$\le \frac{2 \cdot 2^{n^2/2}}{n! \, 2^{\binom{n}{2}}} = 2 \frac{2^{n/2}}{n!} < 1.$$

And so there is some good coloring of K_N! $\square$

Some remarks about this proof:

- Shows $\exists$ a 'good' coloring of $K_{2^{n/2}}$ (i.e. one avoiding blue and red K_n's), but gives no clue how to actually construct such a coloring! and no one knows how to do this

- have bounds

$$2^{n/2} \leq R(n) \leq 4^{n-1}$$

which are pretty far apart! There are modest improvements to these bounds, but these are still essentially all we know!
(Look up Erdős quote about aliens...)

Now let's take a break...

and when we come back

we can explore an application of

Ramsey theory to planar geometry

on today's worksheet by

working in breakout groups...