Math4990: Permutation Pattern Avoidance + Summary of Course

12/15

Ch. 14 of Bóna

Reminders: • Final exam is due today!

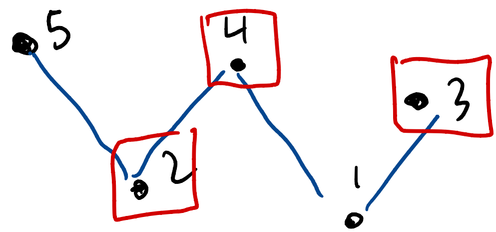
I will try to grade it quickly, so you get your final grade.

Today we'll discuss one more bonus topic, this time in enumerative combinatorics rather than graph theory. The topic is permutation pattern avoidance. Compared to all the math we've seen, this is the newest: with a lot being done in the last ~30 years.

Def'n Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in S_k$ ('one-line notation') be a permutation which we'll call a pattern. We say a bigger permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ contains the pattern σ if there is a subsequence $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ $(i_1 < i_2 < \cdots < i_k)$ whose letters have the same relative order as σ.

E.g. $6\,\underline{1}\,4\,\underline{3}\,2\,\underline{5}$ contains the pattern 123 in the underlined positions. ($a < b < c$) (a, b, c w/)

5 2 4 1 3 contains the pattern 132 in the underlined positions. (a, b, c w/ $a < c < b$)



can be helpful to depict perm's graphically like this to see patterns

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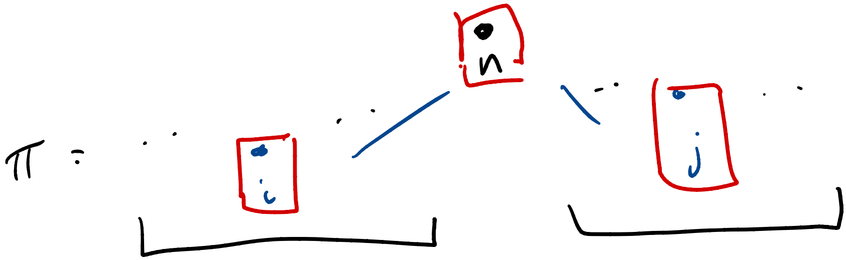
Def'n If $\pi \in S_n$ does not contain a pattern $\sigma \in S_k$, we say π **avoids** σ. We use

$$Av_n(\sigma) := \{\pi \in S_n : \pi \text{ avoids } \sigma\}.$$

Big interest in understanding $Av_n(\sigma)$ for various σ, and counting $\# Av_n(\sigma)$.

Let's consider $\sigma = 132$. What does a **132-avoiding** permutation $\pi \in Av_n(132)$ look like? Notice that the position of the letter n matters a lot!



If we have any $i<j$ w/ i before the n and j after, this would give a 132 pattern.

So all letters before the n must be less than all letters after the n:

$$\underbrace{\overbrace{\qquad\qquad}^{k+1,\dots,n-1 \text{ in some order}}}_{\pi'} \quad n \quad \underbrace{\overbrace{\qquad\qquad}^{1,\dots,k \text{ in some order}}}_{\pi''}$$

Moreover $\pi'' \in S_k$ and $\pi' \in S_{n-1-k}$ also have to be 132-avoiding; e.g:

$$\pi = \underbrace{756}_{\pi'}\ 8\ \underbrace{3412}_{\pi''}$$

And as long as π', π'' are 132-avoiding, π will be as well. So if we define:

$$f(n) := \# Av_n(132),$$

What we just explained implies the recurrence:

$$f(n) = \sum_{i=0}^{n-1} f(i) \cdot f(n-1-i).$$

Do we recall this recurrence anywhere?

Hint: Can compute $f(1), f(2), f(3), \ldots = \underset{\text{all } S_1}{1}, \underset{\text{all } S_2}{2}, \underset{S_3 - 132}{5}, \ldots$

It's the Catalan number recurrence!

Thm $\#\mathrm{Av}_n(132) = C_n = \frac{1}{n+1}\binom{2n}{n}$.

What about $\mathrm{Av}_n(\sigma)$ for other $\sigma \in S_k$?

Def'n The reverse of a perm. $\pi = \pi_1 \cdots \pi_n \in S_n$ is $\pi^{rev} := \pi_n \pi_{n-1} \cdots \pi_1$.

e.g. $132^{rev} = 231$

The complement of $\pi = \pi_1 \cdots \pi_n \in S_n$ is

$$\pi^{co} = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n).$$

e.g. $132^{co} = 312$

Easy prop. π avoids σ $\Leftrightarrow$ π^{rev} avoids σ^{rev}

$\Leftrightarrow$ π^{co} avoids σ^{co}

Cor:

- $\#Av_n(132) = \#Av_n(231) = \#Av_n(312) = \#Av(213)$
- $\#Av_n(123) = \#Av_n(321)$

We know all these are counted by the Catalan #'s. But what about these?

Thm $\#Av_n(123) = \#Av_n(132)\ \forall n$

$\left(= \frac{1}{n+1}\binom{2n}{n}\right)$

Pf: We will create a bijection

$$Av_n(132) \longrightarrow Av_n(123).$$

The bijection is based on the notion of left-to-right minima, which we saw a while ago: recall a LRM of a perm. $\pi = \pi_1 \cdots \pi_n \in S_n$ is a letter π_i less than everything to its left.

e.g. $\pi = \underline{6}\,7\,8\,9\,\underline{5}\,\underline{4}\,10\,\underline{1}\,2\,3 \in Av_{10}(132)$

Claim: $\forall \pi \in Av_n(132)$, there is a unique $\pi' \in Av_n(123)$ with the same LRM in same positions.

e.g. $\pi' = \underline{6}\,9\,8\,7\,\underline{5}\,\underline{4}\,10\,\underline{1}\,3\,2 \in Av_{10}(123)$

How did we make π'? In spaces between LRM we reversed the letters: $789 \mapsto 987$.

Little bit more argument shows $\pi \mapsto \pi'$ is a bijection $Av_n(132) \to Av_n(123)$ (hint: 123 patterns become 132 or 321 patterns)

So indeed we have $\# Av_n(132) = \# A_n(123)$. $\boxtimes$

So we've seen that $\#Av_n(\sigma) = C_n = \frac{1}{n+1}\binom{2n}{n}$ for all patterns $\sigma \in S_3$.

What about for longer patterns?

Already for $\sigma \in S_4$, situation very different.
Up to reverse/complement, three patterns in S_4:

$$1234, \quad 1342, \quad 1324$$

But:

Gessel

$$Av_n(1234) = \frac{1}{(n+1)^2(n+2)}\sum_{k=0}^{n}\binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1}$$

Bóna

$$Av_n(1342) = \frac{(7n^2-3n-2)}{2}\cdot(-1)^{n-1} + 3\sum_{i=2}^{n}2^{i+1}\cdot\frac{(2i-4)!}{i!(i-2)!}\cdot\binom{n-i+2}{2}\cdot(-1)^{n-i}$$

$$Av_n(1324) = ???$$

Zeilberger has said "Not even God knows the number of 1324-avoiders of length 1000".

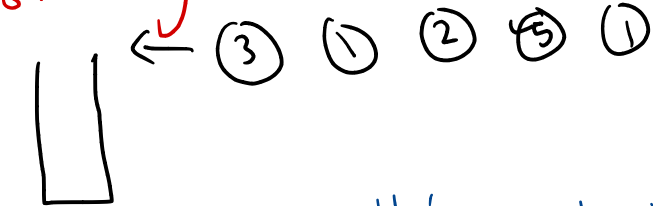
In Ch. 14 of book, can see much more about pattern avoidance, e.g. why

$$Av_n(1342) \le Av_n(1234) \le Av_n(1324) \quad \forall n.$$

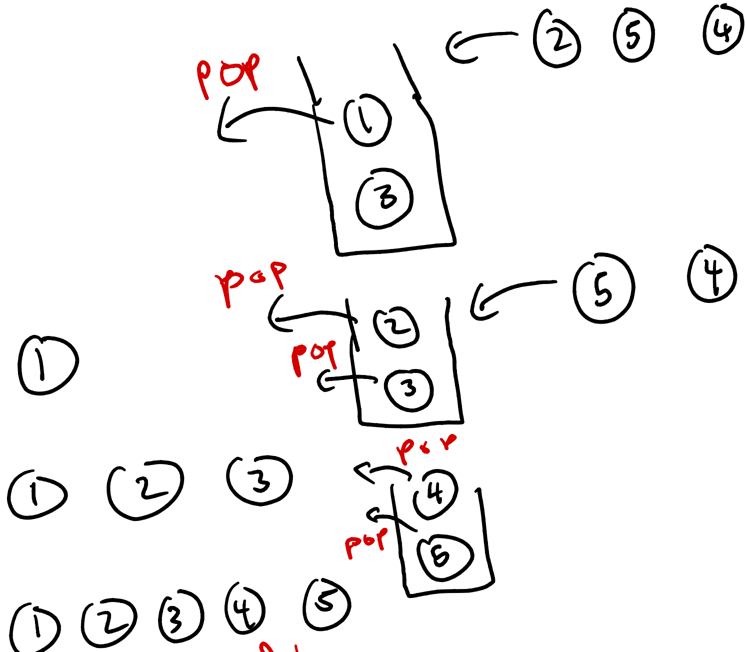
Plus there are lots of open problems in this area!

One more fun instance of perm. patterns:

Stack-sorting a permutation:

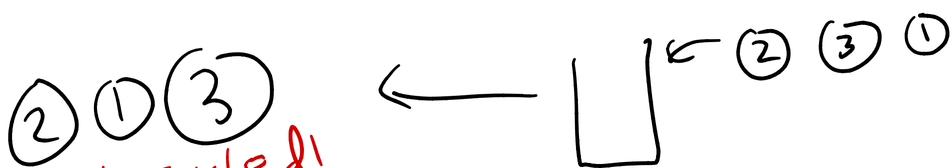


stack (bigger #'s on bottom)



sorted!

Thm π gets sorted by a stack $\iff \pi \in Av_n(231)$



not sorted!

Rather than do a worksheet today, I thought it'd be nice to go over a summary of the course!

- Basic tools:
  - Induction / Pigeonhole principle
  - Principle of Inclusion-Exclusion
  - Generating functions
- Enumeration:
  - sets, subsets (+ Pascal's triangle) binomial coeff's
  - permutations (1^{st} kind Stirling #'s)
  - (set) partitions (2^{nd} kind Stirling #'s)
  - Catalan numbers!
  - Bonus - pattern avoidance
- Graph theory:
  - paths + cycles (Eulerian, hamiltonian, ..)
  - Trees
  - Coloring (bipartite graphs)
  - Matchings
  - planar graphs
  - Bonus - Ramsey theory (probabilistic method)

You've all been wonderful students.

Thank you for the semester +

have a nice (+ safe) winter break!