Math 4990: Partitions, et cetera

9/29

Ch. 5

Reminders: • HW#2 has been posted,
• Should get HW#1 back soon, if not already...

We've discussed basic enumeration problems concerning subsets and words (including permutations), etc. Today we'll continue with problems that are slightly "harder." All of these involve counting ways to break up a whole into parts.

Compositions How many ways are there to distribute 13 (identical) candies to 4 (distinguishable) children?

Same as ways to write 13 as a sum of 4 numbers,

e.g. $13 = \underbrace{4}_{\text{1st kid}} + \underbrace{1}_{\text{2nd kid}} + \underbrace{6}_{\text{3rd}} + \underbrace{2}_{\text{4th}}$

Def'n A **composition** of n into k **parts** is a way of writing n as a sum of k positive integers. If we allow 0 as a part, call it a **weak composition** of n.

each kid gets at least one candy

Prop: # compositions of n into k parts $=$ # weak comp. of $(n-k)$ into k parts.

Pf: $\exists$ bijection which just subtracts one from each part. $\square$

How do we count these?

We actually saw the idea before...

Prop. # weak comp. of n into k parts

$$= \binom{n+k-1}{k-1}$$

Pf: "**stars and bars**" (which we used to count multisets)

$$4 = 0+1+2+0+1 \Longleftrightarrow \underset{0}{\vphantom{*}}|\underset{1}{*}|\underset{2}{**}|\underset{0}{\vphantom{*}}|\underset{1}{*}$$

$\square$

Cor. # comp. of n into k parts $= \binom{n-1}{k-1}$

why?

Cor. # comp. of n into any # of parts $= 2^{n-1}$.

What if children also were indistinguishable?

I.e., how do we count ways to place n indentical balls into k identical boxes?

Def'n A partition of n into k parts is an unordered way of writing n as sum of k positive integers.

E.g. $5 = 2 + 2 + 1$ ($\sim 1 + 2 + 2$)

convention: write parts in decreasing order.

$p_k(n) :=$ # partitions of n into k parts

$p(n) :=$ # partitions of $n = \sum_{k=1}^{n} p_k(n)$

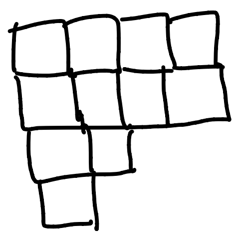
n		$p(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	4, 3+1, 22, 211, 1111	5
5	5, 41, 32, 311, 221, 2111, 11111	7
6		11

In contrast to compositions much harder to understand $p_k(n)$, $p(n)$ in a nice way.

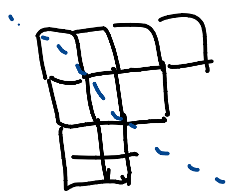
Thm (Well beyond this class) $p(n) \sim \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$

Even if we won't easily be able to count them, let's think a little more about partitions...

$\exists$ a very nice graphical representation of a partition, called its Young diagram:

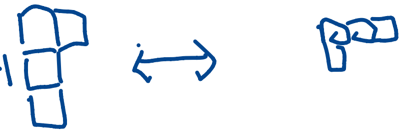
$4+4+2+1 \iff$  $= \lambda$

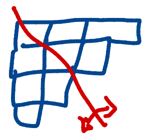
We see a new symmetry from Young diagram: λ partition, its conjugate λ^t has transposed Young diagram:

$4+3+2+2 \iff$  $= \lambda^t$

Prop. $P_k(n)$ ($=$ # partitions of n into k parts)

$=$ # partitions of n w/ largest part k.

Pf: ??? 2+1+1  3+1



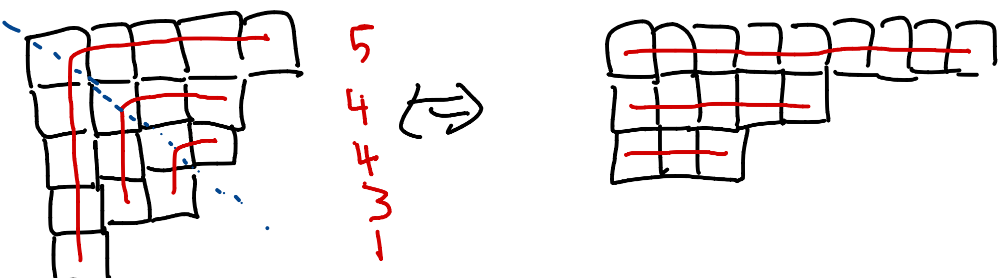
$\blacksquare$

Can we say anything about self-conjugate partitions (i.e., equal to own conjugate)?

Thm. # self-conjugate partitions of n

$=$ # partitions of n into distinct, odd parts.

Pf: Look at this picture:



any self-conjugate partition can be decomposed into "elbows" like this

$\blacksquare$

Have to mention a similar result relating two partition classes...

Thm # partitions of n into odd parts = # partitions of n into distinct parts

E.g. $n=5$

odd = $O(n)$	distinct = $D(n)$
5	5
3+1+1	4+1
1+1+1+1+1	3+2

Pf: $\exists$ bijection $O(n) \to D(n)$ using binary representation!

e.g. odd parts

$$\overset{n}{\underset{\parallel}{26}} = 5+5+5+3+3+1+1+1+1+1$$

$$= 3(5)+2(3)+5(1)$$

binary

$$= (2^1+2^0)5 + (2^1)(3) + (2^2+2^0)(1)$$

$$= 10 + 5 + 6 + 4 + 1$$

$\blacksquare$

distinct parts!

think ab^t later

There are many, many more interesting things to be said about integer partitions (e.g., look up "Euler's pentagonal # theorem") and we (probably) will return to them when we discuss generating functions in a little bit.

But we lost our main focus! ...

Now let's go back to balls and boxes...

what if the balls are distinguishable?

<u>Def'n</u> A **set partition** of $[n] = \{1, 2, \dots, n\}$ is a set $\{P_1, P_2, \dots, P_k\}$ of **parts** (or **blocks**) $P_i \subseteq [n]$ which are:

- **nonempty** $(P_i \neq \emptyset)$
- **pairwise disjoint** $(P_i \cap P_j = \emptyset)$
- their **union** is all of $[n]$. $(\bigcup P_i = [n])$

<u>E.g.</u> $\{\{1,3,4\}, \{2,5\}, \{6,8\}, \{7\}\}$ is a set partition of $[n]$
$\quad\quad$ "
$\{\{5,2\}, \{7\}, \{4,3,1\}, \{6,8\}\}$

$n=8$

$S(n,k) :=$ # partitions of $[n]$ into k parts

$=$ # ways to put n distinct balls into k identical boxes

"Stirling #'s of the 2^{nd} kind"

$$B(n) := \sum_{k=1}^{n} S(n,k) = \text{\# ways to put } n \text{ distinct balls into some \# of identical boxes}$$

"Bell #'s"

E.g. $B(3) = 5$ since: $(123) \mid (12)\ (3) \mid (13)\ (2) \mid (1)\ (23) \mid (1)\ (2)\ (3)$

1 box — 2 boxes — 3 boxes

What if the boxes are distinguishable?

Prop: # ways to put n dist. balls into k dist. boxes $= k! \cdot S(n,k)$.

Pf: $k!$ ways to permute boxes ∎

NOTE! $k! \cdot S(n,k) =$ # surjective functions $f : [n] \to [k]$

Think about why this is for a second...

(Something reminiscent of binomial thm...)

<u>Prop</u>: $x^n = \sum_{k=1}^{n} S(n,k) \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{= (x)_k}$.

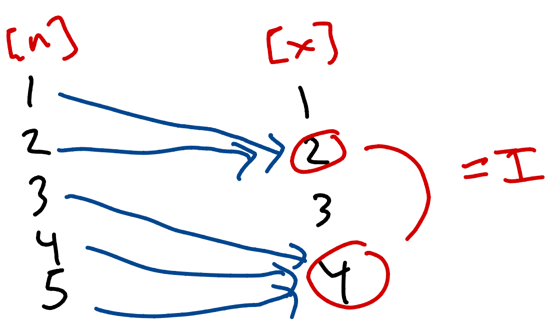
<u>Pf</u>: Let $x \in \mathbb{N}$, so $x^n = \#$ functions $[n] \to [x]$

why is this?

To define $f: [n] \to [x]$:

- choose its image $I \subseteq [x]$, $\# I = k$
- pick a surjection $[n] \to I$.

e.g.



There are $\binom{x}{k}$ choices for 1st item and $k! \cdot S(n,k)$ for 2nd. And then,

$\binom{x}{k} \cdot k! \cdot S(n,k) = S(n,k)\, x(x-1)\cdots(x-k+1)$.

and sum over all possible k. $\blacksquare$

The $S(n,k)$ satisfy an important recurrence relation:

Prop: $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$. $(*)$

Pf: You'll do on worksheet ... ???

$(*)$ implies that $S(n,k)$ are easy to compute (at least, easier than $p_k(n)$) ...

Altogether, for balls and boxes, we have:

	parameters	formula
Surjections	n distinct objects k distinct boxes	$S(n,k)k!$
	n distinct objects any number of distinct boxes	$\sum_{i=1}^{n} S(n,i)i!$
Compositions	n identical objects k distinct boxes	$\binom{n-1}{k-1}$
	n identical objects any number of distinct boxes	2^{n-1}
Set partitions	n distinct objects k identical boxes	$S(n,k)$
	n distinct objects any number of identical boxes	$B(n)$
Integer partitions	n identical objects k identical boxes	$p_k(n)$
	n identical objects any number of identical boxes	$p(n)$

Table 5.1. Enumeration formulae if no boxes are empty.

LOOK up the "12-fold way"

Now let's

take a break...

And when we come back we can do group work on a worksheet where we learn a little bit more about Stirling #'s of the 2^{nd} kind

(+ also maybe preview the Principle of Inclusion-Exclusion!)