11/3

Math 4990: Intro to graph theory - Walks + Paths

Ch. 9 of Bóna

Reminders: - Should get graded HW #3's back soon, if not already...

- HW #4 posted, due in a week (11/10)

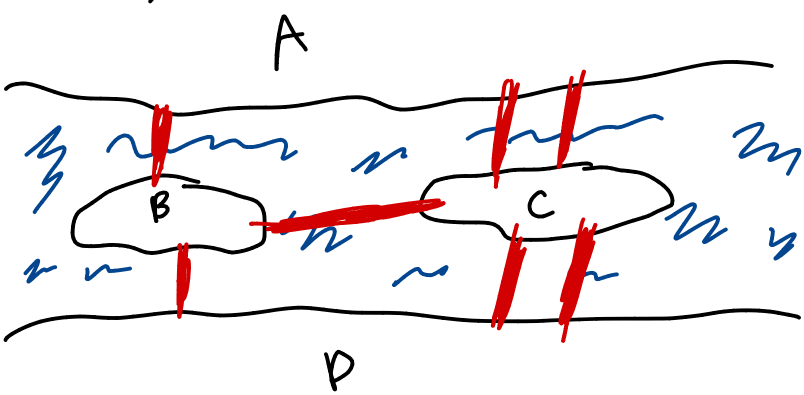
Today we will start discussing graph theory, a topic that we'll explore for most of the rest of the course.

Origin of graph theory: Bridges of Königsberg

Konigsberg was a city in 18^{th} century East Prussia whose unique geography led to the birth of graph theory.

It was on a river and had 7 bridges.

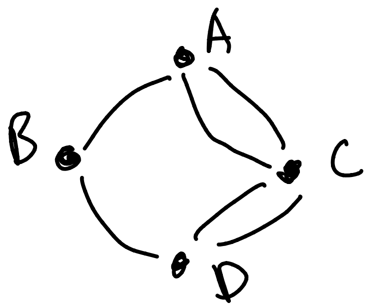
People wondered whether it was possible to cross every bridge exactly once and end up where you started. Here's how the city looked:



The best mathematician of the time was Leonard Euler, and he got interested in this puzzle. He was able to show the answer was no, you can't! And in doing so, he developed a lot of the basics of graph theory.

Euler realized that the exact shapes of the landmasses in Königsberg was irrelevant to the problem, all that mattered was how they were connected. In other words, all the relevant information

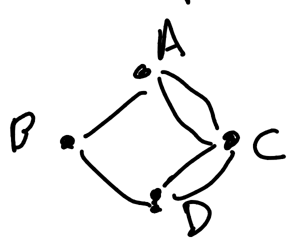
Can be encoded in a diagram like this:



A diagram like this is called a graph.

Formally, a graph $G = (V, E)$ consists of a set of vertices (the points, e.g., A, B, C, D) V and a set of edges (the lines connecting the points) E, where an edge $e \in E$ is an (unordered) pair $e = \{u, v\}$ of vertices $u, v \in V$.

We represent graphs via diagrams like the above, but it's important to note that the same graph can be drawn in multiple ways:



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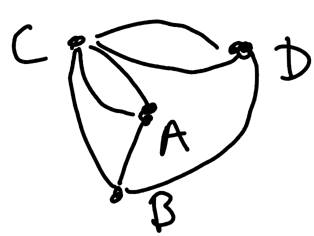


As you can imagine, there are many choices we have for the precise definition of a graph:

- can we have multiple edges between the same pair of vertices, or at most one?
- are we allowed to have a loop, an edge connecting a vertex to itself?

Graphs without multiple edges or loops are called simple graphs. (Note that the bridges of Königsberg graph is <u>not</u> simple.) We will also discuss another variant of graphs, directed graphs, a bit later.

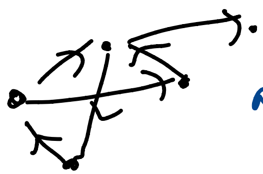
Also worth mentioning that graphs are very versatile data structures. In the B.o.K. problem we see graphs used to represent spatial data (and we'll see later that maps in particular were significant in the development of graph theory), but graphs can be used to represent any (symmetric) relations, e.g., social networks. ← popular application!

Now back to the B.o.K. problem...

Not only did Euler solve this particular bridge problem, but he gave a solution for any possible arrangement of bridges + landmasses.

Some relevant graph terminology:

- a walk in a graph is a way of walking between vertices along edges; more formally it's a sequence $v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$ of vertices v_i and edges e_j such that $e_i = \{v_{i-1}, v_i\}$.



a walk

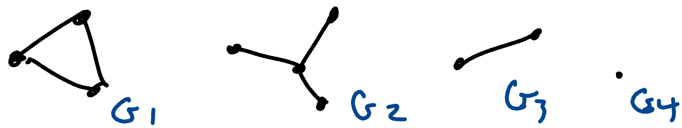
We say that the walk is from v_0 to v_n.

- a closed walk or circuit is a walk from a vertex to itself, i.e., ends where it starts.
- an Eulerian walk (book calls it Eulerian trail) is a walk that uses each edge of graph exactly once.
- an Eulerian circuit is an Eulerian walk that's a circuit.

So the B.o.K. problem is about the existence of Eulerian circuits in graphs, Euler found a simple exact criterion for the existence of an Eulerian circuit in a graph.

To state this criterion, need just two more pieces of graph theory terminology:

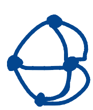
- a graph is connected if there is a walk from any vertex to any other vertex. Every graph is the disjoint union of its connected components:

$G =$ 

G_i = conn. comp.'s

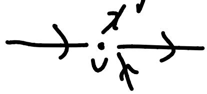
- for $G=(V,E)$ a graph without loops, and $v \in V$ a vertex, the degree of v, denoted $\deg(v)$, is the number of edges containing v. (If G has loops, they count 'double' for degree.)

<u>Thm</u> (Euler) A connected graph $G=(V,E)$ has an Eulerian circuit if and only if every vertex has even degree.

E.g., B.o.K. graph  has degrees 5, 3, 3, 3

$\Rightarrow$ no Eulerian circuit

Pf: (only if direction) In an Eulerian circuit, we walk into any vertex v exactly as often as we walk out of v.

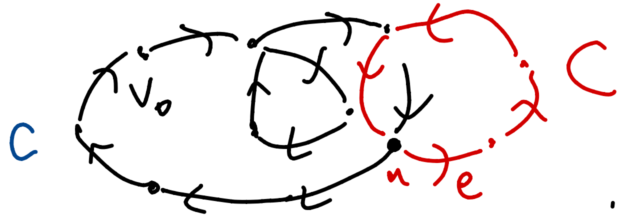


So certainly v must be incident to an even number of edges.

(If direction) Let's construct an Eulerian circuit. Pick any initial vertex $v_0 \in V$. Start walking from v_0: whenever we walk into a vertex v, walk out along an edge we haven't yet traversed. Do this as long as we can: by the even degree assumption, the only place we can stop is at v_0, so we make a circuit C:

If we've used all the edges of G in C then we're done. Otherwise, by connectedness, there must be an edge e leaving a vertex u of C that we didn't use:

(why?)

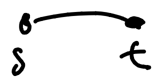


Then, as shown above, let's start walking out from u along that edge, using edges we haven't traversed (including in C). Again, even degrees $\Rightarrow$ we can only get stuck at u, so we get a new circuit C'.

Then we can 'join' C and C': walk from v_0 to u along C, then do C', then walk from u back to v_0 along C. Repeat this process until we use all edges. ∎

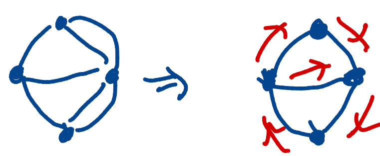
What about if we just want an Eulerian walk?

<u>Thm</u> For $G=(V,E)$ connected, $\exists$ Eulerian walk from s to t w/ $s \neq t \in V$, iff s and t have odd degree and all other vertices have even degree.

<u>Pf:</u> Think about adding edge  ∎

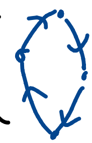
E.g., B.o.K. graph doesn't even have Euler. walk!

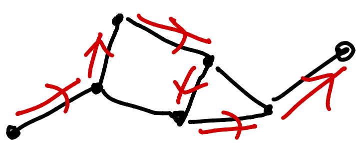
These Eulerian walk/circuit thms are protypical graph theory results characterizing <u>existence</u> of <u>structures</u>.

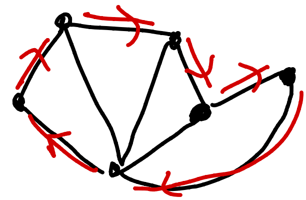
[Aside on history of Königsberg: ]

Hamiltonian paths

Let's discuss another substructure question:

- a path in a graph is a walk that doesn't repeat any vertices: 
- a cycle in a graph is a circuit that doesn't repeat vertices, except start=end 
- a Hamiltonian path is a path that uses every vertex.
- a Hamiltonian cycle is cycle using every vertex:





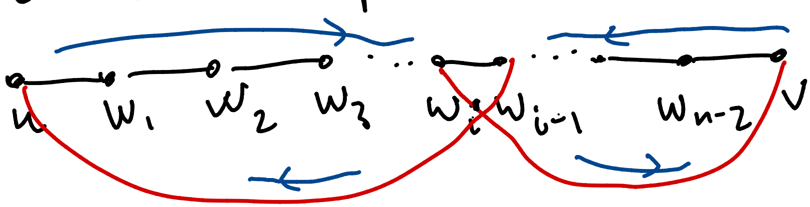
Q: When does a Hamiltonian path/cycle exist in a graph?

A: Much harder to say than for Eulerian walk/circuit: no useful exact criterion (and even hard for a computer!)

Note: adding edges to your graph only improves ability to find Hamilton. path/cycle, so there are are sufficient conditions saying graphs w/ many edges have Hamilton. path/cycles, e.g.:

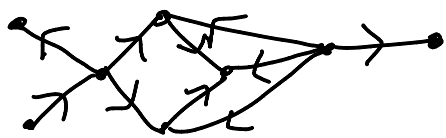
Thm If simple G has n vertices, and every vertex has deg. $\geq n/2$, then it has a Hamilton. cycle.

Pf: Contradiction. Let G be such a graph. Keep adding edges to G until we reach a minimal counterexample G': G' has no Hamilton. cycle, but it would if we add any edge to it. Let u, v be vertices in G' w/out edge $\{u,v\}$. Since adding $\{u,v\}$ makes a Hamilton. cycle, there must be a Hamilton path $u \longrightarrow v$:



Since $\deg(u), \deg(v) \geq n/2$, a Pigeonhole Principle argument says $\exists\, w_i, w_{i+1}$ s.t. $\{u, w_{i+1}\}, \{v, w_i\}$ are edges. But then G' actually has a Hamilton. cycle. $\Rightarrow\Leftarrow$ $\blacksquare$

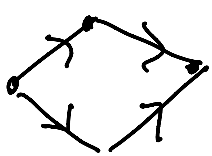
Directed graphs: A directed graph (digraph) D is a variant of a graph where edges come with an orientation $u \to v$. Useful for non-symmetric relations (e.g. 'better than').



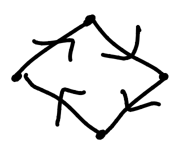
Walks, paths, etc. all make sense for digraphs: we just have to follow direction of edges.

There are 2 notions of connectedness for digraphs:

- D is connected if its underlying undirected graph G is connected. forget orientation of edges
- D is strongly connected if there's a directed walk from any vertex to any other vertex.



connected



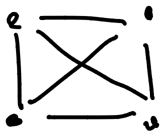
strongly connected

It's also interesting to look for Euler walks/circuits + Hamilton paths/cycles in digraphs.

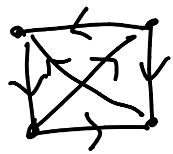
Thm A connected digraph D has an Euler. circuit iff indegree = outdegree $\forall$ vertices.

Pf: Analogous to previous Euler. proof. ∎

The complete graph K_n is the simple undirected graph on n vertices w/ all edges.



K_4



T

A tournament T is an orientation of a complete graph. (Why 'tournament'?)

Easy to see K_n must have Hamilton. path/cycle, but for tournaments T this is interesting:

Thm • Any tournament T has Hamilton. path.

• T has a Hamilton. cycle $\Leftrightarrow$ T is strongly connected.

Pf: See book. Main idea is induction. ∎

Now let's take a break!

And when we come back...

We can explore walks + paths

on today's worksheet in

breakout groups.