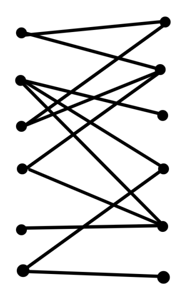
Matchings,
UMTYMP Advanced Topics, Fall 2020

1. Let G be the following bipartite graph on 12 vertices:



   Find a matching of G with the maximum possible number of edges. How do you *know* that this is the maximum?

2. Let G be a bipartite simple graph with bipartition (X, Y). Suppose that $n = |X| = |Y|$ (so the total number of vertices of G is $2n$).

   (a) Suppose G has a perfect matching. Must it be connected?

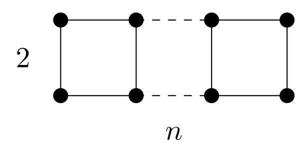
   (b) What is the fewest number of edges G could have if it has a perfect matching.

   (c) Show that G can have $n^2 - n$ edges but still fail to have a perfect matching.

   (d) Can G have $n^2 - n + 1$ edges and fail to have a perfect matching? Explain why not, or give an example. (This question is a bit **tricky**.)

3. How many perfect matchings does the complete bipartite graph $K_{n,n}$ have?

4. Let G be the $2 \times n$ *grid graph*:



   (a) Show that G is bipartite (describe the bipartition (X, Y)).

   (b) Compute the number of perfect matchings of G for $n = 1, 2, 3, 4$.

   (c) Do you recognize this sequence of numbers (we've seen it before!)? Can you prove why, for general n, it is the sequence we've seen before?

Remark: Kasteleyn showed that the number of the number of matchings of the more general $m \times n$ grid graph is:

$$\left(\prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right) \right) \right)^{1/4}.$$

What a crazy formula, huh?! Behind this formula are some *linear algebra* techniques, like the computation of a certain *determinant*. This is kind of similar to, but more advanced than, the Matrix-Tree Theorem we saw for computing the number of spanning trees as a determinant.