Stirling numbers and left-to-right maxima, UMTYMP Advanced Topics, Fall 2020

The *(unsigned) Stirling #'s of the 1st kind* are $c(n,k) := \#p \in S_n$ with k cycles. The *fundamental bijection* $p \mapsto \hat{p}$ writes a permutation in canonical cycle notation and then drops the parentheses and reinterprets it in in one-line notation:

$$p = (4)(521)(76)(83) \mapsto 45217683 = \hat{p}$$

A *left-to-right maxima* of p is a letter that is greater than all letters to its left.

1. Explain why $\sum_{k=1}^{n} c(n,k)x^k = \sum_{p \in S_n} x^{\mathrm{LRMax}(p)}$ using the fundamental bijection, where $\mathrm{LRMax}(p)$ is the number of left-to-right maxima of p.

Imagine building up a permutation $p \in S_n$ in one-line notation step-by-step as follows: first we write down n; then we write down $n-1$ either before or after n; then we write $n-2$ down in one of the 3 "spots" it could occupy; and so on all the way to 1. For example, with $n = 5$:

$$\mathbf{5}$$
$$5\mathbf{4}$$
$$5\mathbf{3}4$$
$$\mathbf{2}534$$
$$2\mathbf{1}534$$

2. Explain why $i \in [n]$ will be a left-to-right maxima in p if and only if when we write i, we write it in the first spot, in this step-by-step procedure building p.

3. Explain why the last item proves

$$\sum_{p \in S_n} x^{\mathrm{LRMax}(p)} = x(x+1)(x+2)\cdots(x+(n-1)).$$

4. What is the generating function $\sum_{p \in S_n} x^{\mathrm{RLMax}(p)}$, where $\mathrm{RLMax}(p)$ is the number of *right-to-left* maxima of p?

5. What is the generating function $\sum_{p \in S_n} x^{\mathrm{LRMin}(p)}$, where $\mathrm{LRMin}(p)$ is the number of left-to-right *minima* of p?